

The spectral theory of the Fourier operator truncated on the positive half-axis.

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Abstract

The spectral analysis of the operator Fourier truncated on the positive half-axis is done.

Notation and terminology.

\mathbb{R} stands for the real axis.

\mathbb{R}^+ stands for the positive real half-axis: $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$.

\mathbb{C} stands for the complex plane.

\mathbb{N} stands for the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \dots\}$

$\mathfrak{M}_{p,q}$, $p \in \mathbb{N}$, $q \in \mathbb{N}$, is the set of all $p \times q$ matrices (p rows, q columns) with entries from \mathbb{C} .

Assume that $(\mathcal{X}, \mathcal{B}, m)$ is a measurable space (\mathcal{B} is a sigma algebra of subsets of \mathcal{X} , m is a non-negative measure defined on σ). If $f(t) : \mathcal{X} \rightarrow \mathbb{R}$ is a \mathcal{B} – *measurable* real valued function on \mathcal{X} , then

$$\operatorname{ess\,sup}_{\mathcal{X}} f(t) = \inf_{\substack{E \in \mathcal{B}: \\ m(E)=0}} \sup_{t \in \mathcal{X} \setminus E} f(t), \quad \operatorname{ess\,inf}_{\mathcal{X}} f(t) = \sup_{\substack{E \in \mathcal{B}: \\ m(E)=0}} \inf_{t \in \mathcal{X} \setminus E} f(t).$$

Assume that the set \mathcal{X} carries two structures: \mathcal{X} is a metric space provided by the metric d , and \mathcal{X} is a measurable space, where m is a non-negative measure defined on the sigma algebra of all Borelian subsets of \mathcal{X} . If a is a point of \mathcal{X} and B is a subset of \mathcal{X} , $B \in \mathcal{B}$, then

$$\operatorname{ess\,dist}(a, B) \stackrel{\text{def}}{=} \operatorname{ess\,inf}_{x \in B} d(a, x).$$

For a set B , $B \in \mathcal{B}$, its essential closure $\text{clos}_e(B)$ is defined as

$$\text{clos}_e(B) = \{a \in \mathbb{X} : \text{ess dist}(a, B) = 0\}.$$

If the set E , $E \in \mathcal{B}$, is of zero measure: $m(E) = 0$, we write $E = \emptyset_e$.

In what follows, the measurable space \mathcal{X} is an interval of a straight line in the complex plane, \mathcal{B} is the sigma-algebra of Borelian subsets of this straight line, and m is the linear (one-dimensional) Lebesgue measure on this \mathcal{B} .

1. The Fourier operator truncated on positive half-axis.

In this paper we study the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$,

$$(\mathcal{F}_{\mathbb{R}^+}x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{it\xi} d\xi, \quad t \in \mathbb{R}^+. \quad (1.1)$$

The operator $\mathcal{F}_{\mathbb{R}^+}$ is considered as an operator acting in the space $L^2(\mathbb{R}^+)$ of all square measurable complex valued functions on \mathbb{R}^+ provided with the scalar product

$$\langle x, y \rangle = \int_{\mathbb{R}^+} x(t) \overline{y(t)} dt.$$

The operator $\mathcal{F}_{\mathbb{R}^+}^*$ adjoint to the operator $\mathcal{F}_{\mathbb{R}^+}$ with respect to this scalar product is

$$(\mathcal{F}_{\mathbb{R}^+}^*x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} x(\xi) e^{-it\xi} d\xi, \quad t \in \mathbb{R}^+. \quad (1.2)$$

The operator $\mathcal{F}_{\mathbb{R}^+}$ is the operator of the form

$$\mathcal{F}_{\mathbb{R}^+} = P_{\mathbb{R}^+} \mathcal{F} P_{\mathbb{R}^+}|_{L^2(\mathbb{R}^+)}, \quad (1.3)$$

where \mathcal{F} is the Fourier operator on the whole real axis:

$$(\mathcal{F}x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x(\xi) e^{it\xi} d\xi, \quad t \in \mathbb{R}, \quad (1.4)$$

$$\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

and $P_{\mathbb{R}^+}$ is the natural orthogonal projector from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}^+)$:

$$(P_{\mathbb{R}^+}x)(t) = \mathbb{1}_{\mathbb{R}^+}(t)x(t), \quad x \in L^2(\mathbb{R}), \quad (1.5)$$

$\mathbb{1}_{\mathbb{R}^+}(t)$ is the indicator function of the set \mathbb{R}^+ . For any set E , its indicator function $\mathbb{1}_E$ is

$$\mathbb{1}_E(t) = \begin{cases} 1, & \text{if } t \in E, \\ 0, & \text{if } t \notin E. \end{cases} \quad (1.6)$$

It should be mention that the Fourier operator \mathcal{F} is an unitary operator in $L^2(\mathbb{R})$:

$$\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = \mathcal{J}_{L^2(\mathbb{R})}, \quad (1.7)$$

$\mathcal{J}_{L^2(\mathbb{R})}$ is the identity operator in $L^2(\mathbb{R})$.

From (1.3) and (1.7) it follows that the operators $\mathcal{F}_{\mathbb{R}^+}$ and $\mathcal{F}_{\mathbb{R}^+}^*$ are contractive: $\|\mathcal{F}_{\mathbb{R}^+}\| \leq 1$, $\|\mathcal{F}_{\mathbb{R}^+}^*\| \leq 1$. We show later that actually

$$\|\mathcal{F}_{\mathbb{R}^+}\| = 1, \quad \|\mathcal{F}_{\mathbb{R}^+}^*\| = 1. \quad (1.8)$$

Nevertheless, these operators are strictly contractive:

$$\|\mathcal{F}_{\mathbb{R}^+}x\| < \|x\|, \quad \|\mathcal{F}_{\mathbb{R}^+}^*x\| < \|x\|, \quad \forall x \in L^2(\mathbb{R}^+), x \neq 0, \quad (1.9)$$

and their spectral radii $r(\mathcal{F}_{\mathbb{R}^+})$ and $r(\mathcal{F}_{\mathbb{R}^+}^*)$ are less that one:

$$r(\mathcal{F}_{\mathbb{R}^+}) = r(\mathcal{F}_{\mathbb{R}^+}^*) = 1/\sqrt{2}. \quad (1.10)$$

In particular, the operators $\mathcal{F}_{\mathbb{R}^+}$ and $\mathcal{F}_{\mathbb{R}^+}^*$ are contractions of the class C_{00} in the sense of [SzNFo].

In [SzNFo], a spectral theory of contractions in a Hilbert space is developed. The starting point of this theory is the representation of the given contractive operator A acting is the Hilbert space \mathcal{H} in the form

$$A = PUP, \quad (1.11)$$

where U is an unitary operator acting is some *ambient* Hilbert space \mathfrak{H} , $\mathcal{H} \subset \mathfrak{H}$, and P is the orthogonal projector from the whole space \mathfrak{H} onto its subspace \mathcal{H} . In the construction of [SzNFo] there is required that not only the equality (1.11) but also the whole series of the equalities

$$A^n = PU^nP, \quad n \in \mathbb{N}, \quad (1.12)$$

hold. The unitary operator U acting in the ambient Hilbert space \mathfrak{H} , $\mathcal{H} \subset \mathfrak{H}$, is said to be *the unitary dilation of the operator A* , $A : \mathcal{H} \rightarrow \mathcal{H}$, if the equalities (1.12) hold. In [SzNfo] it was shown that every contractive operator A admits an unitary dilation. Using the unitary dilation, a functional model of the operator A is constructed. This functional model is an operator acting in some Hilbert space of analytic functions. The functional model of the operator A is an operator which is unitary equivalent to A . The spectral theory of the original operator A is developed by analyzing its functional model.

The relation (1.3) is of the form (1.11), where $\mathcal{H} = L^2(\mathbb{R}^+)$, $\mathfrak{H} = L^2(\mathbb{R})$, $U = \mathcal{F}$, $A = \mathcal{F}_{\mathbb{R}^+}$, $P = P_{\mathbb{R}^+}$ is an orthoprojector from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}^+)$, (1.5). However, for these objects the equalities (1.12) do not hold for all $n \in \mathbb{N}$, but only for $n = 0, 1$. So, the operator \mathcal{F} is not an unitary delation of its truncation $\mathcal{F}_{\mathbb{R}^+}$. Nevertheless, we succeeded in constructing such a functional model of the operator $\mathcal{F}_{\mathbb{R}^+}$ which is easily analyzable. Analyzing this model, we develop the complete spectral theory of the operator $\mathcal{F}_{\mathbb{R}^+}$.

2. Operator calculus for the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$.

In this section, we present the formulations of some of our main results. Proofs will be presented later, in the next sections.

First we set the terminology and notation related to the notion of spectrum of a linear operator.

Let A be a linear operator acting in the Hilbert space \mathcal{H} , with the domain of definition \mathcal{D}_A . (\mathcal{D}_A is assumed to be a linear subspace of \mathcal{H} , not necessary closed.) The *resolvent set* $\rho(A)$ of the operator A is the set of all complex numbers λ for which the operator $\lambda \mathcal{I} - A$ maps one-to-one \mathcal{D}_A onto the whole \mathcal{H} and the inverse operator $(\lambda \mathcal{I} - A)^{-1} : \mathcal{H} \rightarrow \mathcal{D}_A$ is a bounded operator in \mathcal{H} . The complement $\sigma(A) = \mathbb{C} \setminus \rho(A)$ of the resolvent set $\rho(A)$ is said to be the *spectrum of the operator A* .

The resolvent set of every closed operator is an open set (may be empty). If the operator A is a bounded everywhere defined operator, that is $\mathcal{D}_A = \mathcal{H}$, then the resolvent set $\rho(A)$ is not empty: $\rho(A) \supseteq \{\lambda : |\lambda| > \|A\|\}$. The spectrum of a bounded everywhere defined linear operator is a not empty set.

The operator valued function $(z \mathcal{I} - A)^{-1}$ of the variable z defined

on the resolvent set $\rho(A)$ of the operator A is said to be the *resolvent of the operator A* and denoted by $R_A(z)$:

$$R_A(z) = (z\mathcal{I} - A)^{-1}. \quad (2.1)$$

We adhere to the following classification of points of the spectrum of a linear operator. (See [DuSch, VII.5.1].)

Definition 2.1. *Let A be a linear operator in a Hilbert space \mathcal{H} , with the domain of definition \mathcal{D}_A .*

1. *The set of all $\lambda \in \sigma(A)$ such that the mapping $\lambda\mathcal{I} - A : \mathcal{D}_A \rightarrow \mathcal{H}$, is not one-to-one is called the point spectrum of A , and is denoted by $\sigma_p(A)$. Thus, $\lambda \in \sigma_p(A)$ if and only if $Ax = \lambda x$ for some non-zero $x \in \mathcal{D}_A$.*
2. *The set of all $\lambda \in \sigma(A)$ for which the subspace $(\lambda\mathcal{I} - A)\mathcal{D}_A$ is not dense in \mathcal{H} is called the residual spectrum of A , and is denoted by $\sigma_r(A)$. Thus, $\lambda \in \sigma_r(A)$ if and only if $A^*x = \bar{\lambda}x$ for some non-zero $x \in \mathcal{D}_{A^*}$, where A^* is the operator adjoint to the operator A .*
3. *The set of all $\lambda \in \sigma(A)$ for which the mapping $\lambda\mathcal{I} - A : \mathcal{D}_A \rightarrow \mathcal{H}$, is one-to-one, the subspace $(\lambda\mathcal{I} - A)\mathcal{D}_A$ is dense in \mathcal{H} , but the inverse operator $(\lambda\mathcal{I} - A)^{-1} : (\lambda\mathcal{I} - A)\mathcal{D}_A \rightarrow \mathcal{D}_A$ is unbounded is called the continuous spectrum of A , and is denoted by $\sigma_c(A)$. Thus, $\lambda \in \sigma_c(A)$ if and only if $\lambda \notin \sigma_p(A)$, $\lambda \notin \sigma_r(A)$, and there exists a sequence $\{x_n\}_n$ such that*

$$x_n \in \mathcal{D}_A, \quad \|x_n\| = 1, \quad \text{but} \quad \|(A - \lambda\mathcal{I})x_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

If the operator A is closed, in particular if $\mathcal{D}_A = \mathcal{H}$ and A is bounded, then

$$\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A).$$

This classification is rough, sometimes one introduces the more fine classification, like *essential spectrum*, *approximate point spectrum*, etc. However we keep in the classification of Definition 2.1.

Let a and b be points of \mathbb{C} . By definition, the interval $[a, b]$ is the set: $[a, b] = \{(1 - \tau)a + \tau b : \tau \text{ runs over } [0, 1]\}$. The open interval (a, b) , as well as half-open intervals are defined analogously.

1. The spectrum of $\mathcal{F}_{\mathbb{R}^+}$.

Theorem 2.1.

1. The spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ of the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$ is:

$$\sigma(\mathcal{F}_{\mathbb{R}^+}) = [-2^{-1/2} e^{i\pi/4}, 2^{-1/2} e^{i\pi/4}] \quad (2.2)$$

2. The truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$ has no point spectrum and no residual spectrum:

$$\sigma_p(\mathcal{F}_{\mathbb{R}^+}) = \emptyset, \quad \sigma_r(\mathcal{F}_{\mathbb{R}^+}) = \emptyset. \quad (2.3)$$

Thus, its spectrum is continuous:

$$\sigma(\mathcal{F}_{\mathbb{R}^+}) = \sigma_c(\mathcal{F}_{\mathbb{R}^+}).$$

Theorem 2.1*.

1. The spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+}^*)$ of the operator $\mathcal{F}_{\mathbb{R}^+}^*$, which is adjoint to the operator $\mathcal{F}_{\mathbb{R}^+}$, is:

$$\sigma(\mathcal{F}_{\mathbb{R}^+}^*) = [-2^{-1/2} e^{-i\pi/4}, 2^{-1/2} e^{-i\pi/4}] \quad (2.4)$$

2. The operator $\mathcal{F}_{\mathbb{R}^+}^*$ has no point spectrum and no residual spectrum:

$$\sigma_p(\mathcal{F}_{\mathbb{R}^+}^*) = \emptyset, \quad \sigma_r(\mathcal{F}_{\mathbb{R}^+}^*) = \emptyset. \quad (2.5)$$

Thus, its spectrum is continuous:

$$\sigma(\mathcal{F}_{\mathbb{R}^+}^*) = \sigma_c(\mathcal{F}_{\mathbb{R}^+}^*).$$

The point $\zeta = 0$ is in some sense a singular point for operator $\mathcal{F}_{\mathbb{R}^+}$. (See in particular Theorem 9.2 and Remark 2.2). This point splits the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ on two parts: $\sigma^+(\mathcal{F}_{\mathbb{R}^+})$ and $\sigma^-(\mathcal{F}_{\mathbb{R}^+})$.

Definition 2.2.

$$\sigma^+(\mathcal{F}_{\mathbb{R}^+}) \stackrel{\text{def}}{=} \left(0, \frac{1}{\sqrt{2}} e^{i\pi/4}\right], \quad \sigma^-(\mathcal{F}_{\mathbb{R}^+}) \stackrel{\text{def}}{=} \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, 0\right). \quad (2.6)$$

Thus, the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ of the operator $\mathcal{F}_{\mathbb{R}^+}$ splits into the union

$$\sigma(\mathcal{F}_{\mathbb{R}^+}) = \sigma^+(\mathcal{F}_{\mathbb{R}^+}) \cup \sigma^-(\mathcal{F}_{\mathbb{R}^+}) \cup \{0\}. \quad (2.7)$$

2. Growth of the resolvent of the operator $\mathcal{F}_{\mathbb{R}_+}$ near the spectrum.

Let us discuss how the resolvent $(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}$ growth when z approaches the spectrum $\sigma(\mathcal{F}_{\mathbb{R}_+})$. The growth of the resolvent depends on the point $\zeta \in \sigma(\mathcal{F}_{\mathbb{R}_+})$ which z approaches. Roughly speaking, for every fixed $\zeta \in \sigma(\mathcal{F}_{\mathbb{R}_+})$, $\zeta \neq 0$, the norm $\|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}\|$ grows as $C(\zeta)/|z - \zeta|$ as $z \rightarrow \zeta$. Here $C(\zeta)$, $0 < C(\zeta) < \infty$, is a constant with respect to z . However, $C(\zeta) \rightarrow \infty$ as $\zeta \rightarrow 0$. If $\zeta = 0$, that is if $z \rightarrow 0$, then $\|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}\|$ grows as $|z|^{-2}$. Let us formulate the precise result.

Theorem 2.2. *Let ζ be a point of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}_+})$ of the operator $\mathcal{F}_{\mathbb{R}_+}$, and let the point z lie on the normal to the interval $\sigma(\mathcal{F}_{\mathbb{R}_+})$ at the point ζ :*

$$z = \zeta \pm |z - \zeta|e^{i3\pi/4}. \quad (2.8)$$

Then

1. *The resolvent $(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}$ admits the estimate from above:*

$$\|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}\| \leq A(z) \frac{1}{|\zeta|} \cdot \frac{1}{|z - \zeta|}, \quad (2.9)$$

$$\text{where } A(z) = \frac{(2|z|^2 + 1)^{1/2}}{2}.$$

2. *If moreover the condition $|z - \zeta| \leq |\zeta|$ is satisfied, then the resolvent $(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}$ also admits the estimate from below:*

$$A(z) \frac{1}{|\zeta|} \cdot \frac{1}{|z - \zeta|} - B(z)|\zeta||z - \zeta| \leq \|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}\|, \quad (2.10)$$

$$\text{where } A(z) \text{ is the same that in (2.9) and } B(z) = \frac{4}{(2|z|^2 + 1)^{3/2}}.$$

3. *For $\zeta = 0$, the resolvent $(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}$ admits the estimates*

$$2A(z) \frac{1}{|z|^2} - B(z) \leq \|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}\| \leq 2A(z) \frac{1}{|z|^2}, \quad (2.11)$$

where $A(z)$ and $B(z)$ are the same that in (2.9), (2.10), and z is an arbitrary point of the normal.

In particular, if $\zeta \neq 0$, and z tends to ζ along the normal to the interval $\sigma(\mathcal{F}_{\mathbb{R}_+})$, then

$$\|(z\mathcal{J} - \mathcal{F}_{\mathbb{R}_+})^{-1}\| = C(\zeta) \frac{1}{|z - \zeta|} + O(1), \quad (2.12)$$

where

$$C(\zeta) = \frac{\sqrt{1+2|\zeta|^2}}{2|\zeta|}. \quad (2.13)$$

If $\zeta = 0$ and z tends to ζ along the normal to the interval $\sigma(\mathcal{F}_{\mathbb{R}^+})$, then

$$\|(z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}\| = |z|^{-2} + O(1), \quad (2.14)$$

where $O(1)$ is a value which remains bounded as z tends to ζ .

Remark 2.1. From Theorem 9.2 it follows that the operator $\mathcal{F}_{\mathbb{R}^+}$ is not similar to a normal operator. The similarity of the operator $\mathcal{F}_{\mathbb{R}^+}$ to a normal operator is non-compatible with the growth (2.14) of its resolvent.

Remark 2.2. The point $\zeta = 0$ is a distinguished point of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$. Near this point the resolvent of the operator $\mathcal{F}_{\mathbb{R}^+}$ grows faster than near any other point $\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})$. The point $\zeta = 0$ is a spectral singularity. In what follows we still will face a special role of the point $\zeta = 0$ in the spectral theory of the operator $\mathcal{F}_{\mathbb{R}^+}$.

3. The multiplicity of the spectrum of $\mathcal{F}_{\mathbb{R}^+}$.

For operators which are not normal there is no general full-blooded theory of spectral multiplicity. For self-adjoint operators, the property of its spectrum to be of multiplicity one, or in other word the property of the spectrum to be simple, is equivalent to the property of the operator to possess a cyclic vector. Therefore, for an arbitrary operator, the property of the operator to possess a cyclic vector may be accepted as a definition of the simplicity of its spectrum.

We recall that the vector x , $x \in \mathcal{H}$, is said to be cyclic for the operator A , $A : \mathcal{H} \rightarrow \mathcal{H}$, if the linear hull of the set of vectors $\{A^n x\}_{n \in \mathbb{N}}$ is dense in \mathcal{H} .

Theorem 2.3. The spectrum of the operator $\mathcal{F}_{\mathbb{R}^+}$ is simple: there exist vectors which are cyclic for $\mathcal{F}_{\mathbb{R}^+}$.

4. Operator calculus for the operator $\mathcal{F}_{\mathbb{R}^+}$.

Holomorphic operator calculus. We say that a function h is holomorphic on a closed set σ , $\sigma \in \mathbb{C}$, if the function f is defined and holomorphic in an open neighborhood of the set σ . (The neighborhood may depend on the function f .) The set of functions holomorphic

on the set σ forms an algebra over the field of complex numbers. This algebra is denoted by $\text{hol}(\sigma)$.

According to the general theory of linear operators, for arbitrary function h which is holomorphic on the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ of the operator $\mathcal{F}_{\mathbb{R}^+}$ one can define the operator $h(\mathcal{F}_{\mathbb{R}^+})$ by means of the integral

$$h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\Gamma} h(z) R_{\mathcal{F}_{\mathbb{R}^+}}(z) dz, \quad (2.15)$$

where $R_{\mathcal{F}_{\mathbb{R}^+}}(z) = (z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}$ is the resolvent of the operator $\mathcal{F}_{\mathbb{R}^+}$, Γ is an arbitrary simple contour which encloses the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ and is contained in the domain of holomorphy of the function h . The integral is taken counterclockwise. The value of this integral does not depend on the contour Γ .

The operator $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+})$ is called *the function h of the operator $\mathcal{F}_{\mathbb{R}^+}$* . The correspondence

$$h(z) \rightarrow h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}), \quad \text{where } h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) \text{ is defined by (2.15),}$$

is said to be the *holomorphic functional calculus for the operator $\mathcal{F}_{\mathbb{R}^+}$* .

The holomorphic functional calculus is a homomorphism of the algebra $\text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$ into the algebra of bounded operators in $\mathcal{H} = L^2(\mathbb{R}^+)$:

Algebraic properties of the holomorphic functional calculus:

1. If $h(\zeta) \equiv 1$, then $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = \mathcal{J}$.
2. If $h(\zeta) \equiv \zeta$, then $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = \mathcal{F}_{\mathbb{R}^+}$.
3. If $h(\zeta) = \alpha_1 h_1(\zeta) + \alpha_2 h_2(\zeta)$, where $h_1, h_2 \in \text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$, $\alpha_1, \alpha_2 \in \mathbb{C}$, then $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = \alpha_1 (h_1)_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) + \alpha_2 (h_2)_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+})$.
4. If $h(\zeta) = h_1(\zeta) \cdot h_2(\zeta)$, where $h_1, h_2 \in \text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$, then $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = (h_1)_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) \cdot (h_2)_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+})$.

From 1-4 it follows

5. If $h(\zeta) \in \text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$, and $h(\zeta) \neq 0$ for $\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})$, $h^{-1}(\zeta) \in \text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$, then the operator $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+})$ is invertible, and $(h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}))^{-1} = (h^{-1})_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+})$.

In particular,

6. If $h(\zeta) \equiv (z - \zeta)^{-1}$, where $z \in \mathbb{C} \setminus \sigma(\mathcal{F}_{\mathbb{R}^+})$, then $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = (z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1}$.

The holomorphic functional calculus is applicable to an arbitrary bounded operator A in a Hilbert space. However the operator $\mathcal{F}_{\mathbb{R}^+}$ is the very specific operator. For this operator, it is possible to define functions $h(\mathcal{F}_{\mathbb{R}^+})$ for functions h from much more wider class than the class $\text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$.

$\mathcal{F}_{\mathbb{R}^+}$ -admissible functions. The next notion is one of the main notions of this work.

Definition 2.3. A function $h(\zeta)$ is said to be $\mathcal{F}_{\mathbb{R}^+}$ -admissible if

1. $h(\zeta)$ is a Borel-measurable function which is defined almost everywhere with respect to the Lebesgue measure on the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+}) = \left[-\frac{1}{\sqrt{2}}e^{i\pi/4}, \frac{1}{\sqrt{2}}e^{i\pi/4} \right]$ of the operator $\mathcal{F}_{\mathbb{R}^+}$.
2. The norm $\|h\|_{\mathcal{F}_{\mathbb{R}^+}}$ is finite, where

$$\|h\|_{\mathcal{F}_{\mathbb{R}^+}} \stackrel{\text{def}}{=} \text{ess sup}_{\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})} \left(\frac{|h(\zeta) + h(-\zeta)|}{2} + \frac{|h(\zeta) - h(-\zeta)|}{2|\zeta|} \right). \quad (2.16)$$

The set of all $\mathcal{F}_{\mathbb{R}^+}$ -admissible functions provided by natural 'pointwise' algebraic operation and the norm (2.16) is denoted by $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$.

An analogous definition related to the adjoint operator $\mathcal{F}_{\mathbb{R}^+}^*$ is:

Definition 2.4. A function $h(\zeta)$ is said to be $\mathcal{F}_{\mathbb{R}^+}^*$ -admissible if

1. $h(\zeta)$ is a Borel-measurable function which is defined almost everywhere with respect to the Lebesgue measure on the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+}^*) = \left[-\frac{1}{\sqrt{2}}e^{-i\pi/4}, \frac{1}{\sqrt{2}}e^{-i\pi/4} \right]$ of the operator $\mathcal{F}_{\mathbb{R}^+}^*$.
2. The norm $\|h\|_{\mathcal{F}_{\mathbb{R}^+}^*}$ is finite, where

$$\|h\|_{\mathcal{F}_{\mathbb{R}^+}^*} \stackrel{\text{def}}{=} \text{ess sup}_{\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+}^*)} \left(\frac{|h(\zeta) + h(-\zeta)|}{2} + \frac{|h(\zeta) - h(-\zeta)|}{2|\zeta|} \right). \quad (2.17)$$

The set of all $\mathcal{F}_{\mathbb{R}^+}^*$ -admissible functions provided by natural "point-wise" algebraic operation and the norm (2.16) is denoted by $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}^*}$.

Lemma 2.1. *Each of two sets $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$, $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}^*}$ is a Banach algebra, and*

$$\|h_1 h_2\|_{\mathcal{F}_{\mathbb{R}^+}} \leq \|h_1\|_{\mathcal{F}_{\mathbb{R}^+}} \|h_2\|_{\mathcal{F}_{\mathbb{R}^+}} \quad \text{or} \quad \|h_1 h_2\|_{\mathcal{F}_{\mathbb{R}^+}^*} \leq \|h_1\|_{\mathcal{F}_{\mathbb{R}^+}^*} \|h_2\|_{\mathcal{F}_{\mathbb{R}^+}^*} \quad (2.18)$$

for every $h_1, h_2 \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$ or $h_1, h_2 \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}^}$ respectively.*

Lemma 2.2. *Let $h(\zeta)$ be an $\mathcal{F}_{\mathbb{R}^+}$ -admissible function.*

The function $h^{-1}(\zeta)$ is an $\mathcal{F}_{\mathbb{R}^+}$ -admissible function if and only if the set of values of the function h is separated from zero, that is the following condition

$$\operatorname{ess\,inf}_{\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})} |h(\zeta)| > 0. \quad (2.19)$$

holds.

If the condition (2.19) holds, then

$$\|h^{-1}(\zeta)\|_{\mathcal{F}_{\mathbb{R}^+}} \leq \|h(\zeta)\|_{\mathcal{F}_{\mathbb{R}^+}} \cdot \left(\operatorname{ess\,inf}_{\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})} |h(\zeta)| \right)^{-2}. \quad (2.20)$$

Definition 2.5. *If $h(\zeta)$ is a complex-valued function defined on a subset S of the complex plane, then the conjugated function $\bar{h}(\zeta)$ is a function defined on the conjugated set \bar{S} by the equality*

$$\bar{h}(\zeta) \stackrel{\text{def}}{=} \overline{h(\bar{\zeta})}. \quad (2.21)$$

Lemma 2.3. *A function h belongs to the algebra $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$ if and only if the conjugate function \bar{h} belongs to the algebra $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}^*}$. Moreover,*

$$\|h\|_{\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}} = \|\bar{h}\|_{\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}^*}}.$$

The resolvent-based functional calculus for the operator $\mathcal{F}_{\mathbb{R}^+}$.

The following theorem is one of the main results of this work:

Theorem 2.4. *Let $h(\zeta)$ be an $\mathcal{F}_{\mathbb{R}^+}$ -admissible function: $h(\zeta) \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$. Then there exists the strong limit*

$$\begin{aligned} h(\mathcal{F}_{\mathbb{R}^+}) &\stackrel{\text{def}}{=} \\ &= \lim_{\varepsilon \rightarrow +0} \frac{1}{2\pi i} \int_{\sigma(\mathcal{F}_{\mathbb{R}^+})} h(\zeta) \left(R_{\mathcal{F}_{\mathbb{R}^+}}(\zeta - \varepsilon i e^{i\pi/4}) - R_{\mathcal{F}_{\mathbb{R}^+}}(\zeta + \varepsilon i e^{i\pi/4}) \right) d\zeta, \end{aligned} \quad (2.22)$$

where $R_{\mathcal{F}_{\mathbb{R}^+}}(z) = (z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1}$ is the resolvent of the operator $\mathcal{F}_{\mathbb{R}^+}$, and the integral is taken along the interval $\sigma(\mathcal{F}_{\mathbb{R}^+}) = \left[-\frac{1}{\sqrt{2}}e^{i\pi/4}, \frac{1}{\sqrt{2}}e^{i\pi/4}\right]$ from the point $-\frac{1}{\sqrt{2}}e^{i\pi/4}$ to the point $\frac{1}{\sqrt{2}}e^{i\pi/4}$.

The following result supplements Theorem 2.4.

Theorem 2.4^s. *Let the function h , which appears in Theorem 2.4, satisfy the extra conditions:*

1. $h(\zeta)$ is continuous for $\zeta \in \left[-\frac{1}{\sqrt{2}}e^{i\pi/4}, \frac{1}{\sqrt{2}}e^{i\pi/4}\right]$;
2. $\frac{h(\zeta) - h(-\zeta)}{\zeta}$ is continuous for $\left[-\frac{1}{\sqrt{2}}e^{i\pi/4}, \frac{1}{\sqrt{2}}e^{i\pi/4}\right]$;
3. $h\left(-\frac{1}{\sqrt{2}}e^{i\pi/4}\right) = 0$, $h\left(\frac{1}{\sqrt{2}}e^{i\pi/4}\right) = 0$.

Then the limit in (2.22) exists in the uniform operator topology.

Definition 2.6. Let $h(\zeta)$ be an $\mathcal{F}_{\mathbb{R}^+}$ -admissible function: $h(\zeta) \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$. The operator $h(\mathcal{F}_{\mathbb{R}^+})$ defined by (2.22) is called the function h of the operator $\mathcal{F}_{\mathbb{R}^+}$.

The correspondence

$$h(z) \rightarrow h(\mathcal{F}_{\mathbb{R}^+}), \quad \text{where } h(\mathcal{F}_{\mathbb{R}^+}) \text{ is defined by (2.22),}$$

is said to be the resolvent-based functional calculus for the operator $\mathcal{F}_{\mathbb{R}^+}$.

Properties of the resolvent-based functional calculus.

Lemma 2.4. *The resolvent-based calculus extends the holomorphic functional calculus:*

1. The algebra $\text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$ is contained in the algebra $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$.
2. For $h \in \text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$, both definitions of the function h of the operator of the operator $\mathcal{F}_{\mathbb{R}^+}$, the definition (2.15) and the definition (2.22), yield the same result, i.e. the integral in the right hand side of (2.15) coincides with the limit of the integrals in the right hand side of (2.22):

$$h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = h(\mathcal{F}_{\mathbb{R}^+}). \quad (2.23)$$

Theorem 2.5. *The resolvent-based functional calculus is a homomorphism of the algebra $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$ of $\mathcal{F}_{\mathbb{R}^+}$ -admissible functions into the algebra of bounded operators in $\mathcal{H} = L^2(\mathbb{R}^+)$:*

1. *If $h(\zeta) \equiv 1$, then $h(\mathcal{F}_{\mathbb{R}^+}) = \mathcal{I}$.*
2. *If $h(\zeta) \equiv \zeta$, then $h(\mathcal{F}_{\mathbb{R}^+}) = \mathcal{F}_{\mathbb{R}^+}$.*
3. *If $h(\zeta) = \alpha_1 h_1(\zeta) + \alpha_2 h_2(\zeta)$, where $h_1, h_2 \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$, $\alpha_1, \alpha_2 \in \mathbb{C}$, then $h(\mathcal{F}_{\mathbb{R}^+}) = \alpha_1 h_1(\mathcal{F}_{\mathbb{R}^+}) + \alpha_2 h_2(\mathcal{F}_{\mathbb{R}^+})$.*
4. *If $h(\zeta) = h_1(\zeta) \cdot h_2(\zeta)$, where $h_1, h_2 \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$, then $h(\mathcal{F}_{\mathbb{R}^+}) = h_1(\mathcal{F}_{\mathbb{R}^+}) \cdot h_2(\mathcal{F}_{\mathbb{R}^+})$.*
5. *If $h \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$, and $h^{-1} \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$ (see Lemma 2.2), then the operator $h(\mathcal{F}_{\mathbb{R}^+})$ is invertible, and*

$$(h(\mathcal{F}_{\mathbb{R}^+}))^{-1} = (h^{-1})(\mathcal{F}_{\mathbb{R}^+}).$$

Theorem 2.6. *The two-sides estimate*

$$\frac{1}{2} \|h\|_{\mathcal{F}_{\mathbb{R}^+}} \leq \|(h(\mathcal{F}_{\mathbb{R}^+}))\| \leq \|h\|_{\mathcal{F}_{\mathbb{R}^+}}, \quad (2.24)$$

holds for every function $h \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$, where $\|h(\mathcal{F}_{\mathbb{R}^+})\|$ is the norm of the operator $h(\mathcal{F}_{\mathbb{R}^+})$ considered as an operator from $L^2(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$, and the norm $\|h\|_{\mathcal{F}_{\mathbb{R}^+}}$ of the function h is defined in Definition 2.3.

Theorem 2.7. *Let $\{h_n\}_{n \in \mathbb{N}}$ be a sequence of functions from $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$ which satisfies the conditions:*

1. *The norms $\|h_n\|_{\mathcal{F}_{\mathbb{R}^+}}$ are uniformly bounded:*

$$\sup_{n \in \mathbb{N}} \|h_n\|_{\mathcal{F}_{\mathbb{R}^+}} < \infty. \quad (2.25)$$

2. *For m -almost every $\zeta \in \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4}\right]$, there exist the limit*

$$h(\zeta) = \lim_{n \rightarrow \infty} h_n(\zeta). \quad (2.26)$$

Then $h \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$, and

$$h(\mathcal{F}_{\mathbb{R}^+}) = \lim_{n \rightarrow \infty} h_n(\mathcal{F}_{\mathbb{R}^+}), \quad (2.27)$$

where the limit stands for the strong convergence of a sequence of operators.

The next result is a spectral mapping theorem for the $\mathcal{F}_{\mathbb{R}^+}$ -admissible functional calculus. Given a Borelian-measurable complex-valued function h , $h : \sigma(\mathcal{F}_{\mathbb{R}^+}) \rightarrow \mathbb{C}$, the *essential h -image* $(h(\sigma(\mathcal{F}_{\mathbb{R}^+}))_e$ of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ is defined as

$$(h(\sigma(\mathcal{F}_{\mathbb{R}^+}))_e \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{ess\,inf}_{\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})} |z - h(\zeta)| = 0\}, \quad (2.28)$$

where the interval $\sigma(\mathcal{F}_{\mathbb{R}^+})$ is provided by the one-dimensional Lebesgue measure m . (The essential h -image $(h(\sigma(\mathcal{F}_{\mathbb{R}^+}))_e$) is determined by the mapping h rather by the set $h(\sigma(\mathcal{F}_{\mathbb{R}^+}))$).

Theorem 2.8. *Let h be an $\mathcal{F}_{\mathbb{R}^+}$ -admissible function. Then the spectrum $\sigma(h(\mathcal{F}_{\mathbb{R}^+}))$ of the operator $h(\mathcal{F}_{\mathbb{R}^+})$ coincides with the essential h -image of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$:*

$$\sigma(h(\mathcal{F}_{\mathbb{R}^+})) = (h(\sigma(\mathcal{F}_{\mathbb{R}^+}))_e. \quad (2.29)$$

If $z \notin \sigma(h(\mathcal{F}_{\mathbb{R}^+}))$, then the resolvent $R_{h(\mathcal{F}_{\mathbb{R}^+})}(z) = (z\mathcal{I} - h(\mathcal{F}_{\mathbb{R}^+}))^{-1}$ of the operator $h(\mathcal{F}_{\mathbb{R}^+})$ is:

$$R_{h(\mathcal{F}_{\mathbb{R}^+})}(z) = r(\mathcal{F}_{\mathbb{R}^+}), \quad (2.30a)$$

where

$$r(\zeta) = (z - h(\zeta))^{-1}. \quad (2.30b)$$

Proofs of the results formulated in this section will be presented in next sections.

3. Spectral projectors which correspond to the operator $\mathcal{F}_{\mathbb{R}^+}$.

Though the operators \mathcal{F}_E and \mathcal{F}_E^* are non-normal, the $\mathcal{F}_{\mathbb{R}^+}$ -admissible functional calculus allows to some extent to work with these operators as if they are self-adjoint. In particular we construct objects which may be considered as resolutions of identity related to the operators \mathcal{F}_E and \mathcal{F}_E^* . The resolution of identity related to the operator \mathcal{F}_E is a family of its spectral projectors. We construct this family of spectral projectors as the family of functions of the operator \mathcal{F}_E which functions

are the indicator functions of subsets of the spectrum $\sigma_{\mathcal{F}_E}$. Though this family of subsets is not so rich as in the case of self-adjoint operator and does not contain *all* Borelian subsets of $\sigma_{\mathcal{F}_E}$, it is rich enough for our goal.

Definition 3.1. For a subset Δ of the complex plane, we define its symmetric part Δ_s and asymmetric part Δ_a :

$$\Delta_s = \Delta \cap (-\Delta), \quad \Delta_a = \Delta \setminus (-\Delta). \quad (3.1)$$

Here, as usual, $-\Delta = \{z \in \mathbb{C} : -z \in \Delta\}$. So,

$$\Delta = \Delta_s \cup \Delta_a, \quad \Delta_s \cap \Delta_a = \emptyset, \quad \Delta_s = -\Delta_s, \quad \Delta_a \cap (-\Delta_a) = \emptyset. \quad (3.2)$$

With every subset of $\Delta \in \mathbb{C}$, we associate its indicator function $\mathbb{1}_\Delta(z)$:

$$\mathbb{1}_\Delta(z) = 1 \text{ if } z \in \Delta, \quad \mathbb{1}_\Delta(z) = 0 \text{ if } z \notin \Delta.$$

Definition 3.2. The set S , $S \subseteq \sigma_{\mathcal{F}_E}$ is essentially separated from zero if

$$\text{ess dist}(S, 0) > 0.$$

Lemma 3.1. Let Δ be a subset of the spectrum $\sigma_{\mathcal{F}_E}$ of the operator \mathcal{F}_E . The indicator function $\mathbb{1}_\Delta$ is \mathcal{F}_E -admissible if and only if the asymmetric part Δ_a of the set Δ is essentially separated from zero.

Proof. Since the function $\mathbb{1}_\Delta(\zeta)$ is bounded (either $|\mathbb{1}_\Delta(\zeta)| = 1$ or $|\mathbb{1}_\Delta(\zeta)| = 0$), the function $\mathbb{1}_\Delta(\zeta)$ is \mathcal{F}_E -admissible if and only if the function $\frac{\mathbb{1}_\Delta(\zeta) - \mathbb{1}_\Delta(-\zeta)}{\zeta}$ is essentially bounded. In view of (3.2), $\mathbb{1}_\Delta(\zeta) - \mathbb{1}_\Delta(-\zeta) = \mathbb{1}_{\Delta_a}(\zeta) - \mathbb{1}_{\Delta_{-a}}(-\zeta)$. From the other hand,

$$|\mathbb{1}_{\Delta_a}(\zeta) - \mathbb{1}_{\Delta_{-a}}(-\zeta)| = \mathbb{1}_{\Delta_a \cup (-\Delta_a)}(\zeta),$$

so the function $\frac{\mathbb{1}_\Delta(\zeta) - \mathbb{1}_\Delta(-\zeta)}{\zeta}$ is essentially bounded if and only if the function $\frac{\mathbb{1}_{\Delta_a \cup (-\Delta_a)}(\zeta)}{\zeta}$ is essentially bounded. The last function is essentially bounded if and only if the set $\Delta_a \cup (-\Delta_a)$ is essentially separated from zero. From the structure of the set $\Delta_a \cup (-\Delta_a)$ it is clear that the set $\Delta_a \cup (-\Delta_a)$ is essentially separated from zero if and only if the set Δ_a is essentially separated from zero. \square

Definition 3.3. The Borelian subset Δ of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ is said to be $\mathcal{F}_{\mathbb{R}^+}$ -admissible set if the indicator function $\mathbb{1}_\Delta(\zeta)$ of Δ is a $\mathcal{F}_{\mathbb{R}^+}$ -admissible function.

Lemma 3.2. Let Δ_1 and Δ_2 be $\mathcal{F}_{\mathbb{R}^+}$ -admissible subsets of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$. Then the sets $\Delta_1 \cup \Delta_2$, $\Delta_1 \cap \Delta_2$ and $\Delta_1 \setminus \Delta_2$ are $\mathcal{F}_{\mathbb{R}^+}$ -admissible sets as well. In particular if the set Δ is $\mathcal{F}_{\mathbb{R}^+}$ -admissible, then its complement, the set $\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta$, is $\mathcal{F}_{\mathbb{R}^+}$ -admissible as well.

Definition 3.4. Let Δ be a $\mathcal{F}_{\mathbb{R}^+}$ -admissible subset of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$. The operator $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ is defined as

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) \stackrel{\text{def}}{=} \mathbb{1}_\Delta(\mathcal{F}_{\mathbb{R}^+}), \quad (3.3)$$

where $\mathbb{1}_\Delta(\zeta)$ is the indicator function of the set Δ and the function $\mathbb{1}_\Delta(\mathcal{F}_{\mathbb{R}^+})$ of the operator $\mathcal{F}_{\mathbb{R}^+}$ is understood in the sense of Definition 2.6.

Theorem 3.1. The family of the operators $\{\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\}_\Delta$, where Δ runs over the family of all $\mathcal{F}_{\mathbb{R}^+}$ -admissible sets, possesses the following properties:

1. If the sets Δ_1 and Δ_2 are $\mathcal{F}_{\mathbb{R}^+}$ -admissible, then

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1 \cap \Delta_2) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1) \cdot \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_2); \quad (3.4)$$

In particular, for every $\mathcal{F}_{\mathbb{R}^+}$ -admissible set Δ , the operator $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ is a projector¹

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}^2(\Delta) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta); \quad (3.5)$$

2. The projectors corresponding to the $\mathcal{F}_{\mathbb{R}^+}$ -admissible sets \emptyset and $\sigma(\mathcal{F}_{\mathbb{R}^+})$ are:

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\emptyset) = 0; \quad \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+})) = \mathcal{I}, \quad (3.6)$$

where \mathcal{I} is the identity operator in the space $L^2(\mathbb{R}^+)$.

¹The projector $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ may be not orthogonal. See Theorem 3.2 below.

3. The correspondence $\Delta \rightarrow \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)L^2(\mathbb{R}^+)$ between subsets of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ and subspaces of the space $L^2(\mathbb{R}^+)$ preserves the order:

$$\text{If } \Delta_1 \subset \Delta_2, \text{ then } \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1)L^2(\mathbb{R}^+) \subseteq \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_2)L^2(\mathbb{R}^+). \quad (3.7)$$

4. If the sets Δ_1 and Δ_2 are $\mathcal{F}_{\mathbb{R}^+}$ -admissible, and $\Delta_1 \cap \Delta_2 = \emptyset$, then

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1 \cup \Delta_2) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1) + \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_2); \quad (3.8a)$$

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1) \cdot \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_2) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_2) \cdot \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1) = 0. \quad (3.8b)$$

In particular, for every $\mathcal{F}_{\mathbb{R}^+}$ -admissible set Δ , the equalities

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) + \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta) = \mathcal{I}; \quad (3.9a)$$

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) \cdot \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta) \cdot \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) = 0 \quad (3.9b)$$

hold.

Proof. The mapping $\Delta \rightarrow \mathbb{1}_{\Delta}(\zeta)$ possesses the properties:

$$\begin{aligned} \mathbb{1}_{\Delta_1 \cap \Delta_2}(\zeta) &= \mathbb{1}_{\Delta_1}(\zeta) \cdot \mathbb{1}_{\Delta_2}(\zeta) \text{ for every } \Delta_1, \Delta_2, \\ \mathbb{1}_{\Delta_1 \cup \Delta_2}(\zeta) &= \mathbb{1}_{\Delta_1}(\zeta) + \mathbb{1}_{\Delta_2}(\zeta) \text{ if } \Delta_1 \cap \Delta_2 = \emptyset, \\ \mathbb{1}_{\emptyset}(\zeta) &\equiv 0, \text{ and } \mathbb{1}_{\sigma(\mathcal{F}_{\mathbb{R}^+})}(\zeta) \equiv 1 \text{ for } \zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+}). \end{aligned}$$

Statements 1,3,4 of the present Theorem are consequences of these properties of the mapping $\Delta \rightarrow \mathbb{1}_{\Delta}(\zeta)$ and the properties of the mapping $\mathbb{1}_{\Delta}(\zeta) \rightarrow \mathbb{1}_{\Delta}(\mathcal{F}_E)$, which are particular cases of the properties formulated as Statements 1-2 of the Theorem 7.1. The property $\mathcal{P}_{\mathcal{F}_E}(\emptyset) = 0$ is evident. The property $\mathcal{P}_{\mathcal{F}_E}(\sigma(\mathcal{F}_{\mathbb{R}^+})) = \mathcal{I}$, that is the equality $\mathbb{1}_{\sigma(\mathcal{F}_{\mathbb{R}^+})}(\mathcal{F}_{\mathbb{R}^+}) = \mathcal{I}$ is a consequence of the property 1 of the holomorphic functional calculus. The function $\mathbb{1}_{\sigma(\mathcal{F}_{\mathbb{R}^+})}(\zeta)$ can be considered as the restriction of the function $h(\zeta) \equiv 1$, $h \in \text{hol}(\sigma(\mathcal{F}_{\mathbb{R}^+}))$, on the set $\sigma(\mathcal{F}_{\mathbb{R}^+})$. The property $h(\mathcal{F}_{\mathbb{R}^+}) = \mathcal{I}$ is the property 1 of the holomorphic operator calculus. According to the holomorphic functional calculus, $(\mathbb{1}_{\sigma(\mathcal{F}_{\mathbb{R}^+})})_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = \mathcal{I}$. According to Lemma 2.4, $(\mathbb{1}_{\sigma(\mathcal{F}_{\mathbb{R}^+})})_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = \mathbb{1}_{\sigma(\mathcal{F}_{\mathbb{R}^+})}(\mathcal{F}_{\mathbb{R}^+})$. \square

Theorem 3.2.

1. If the set $\Delta, \Delta \subseteq \sigma_{\mathcal{F}_{\mathbb{R}^+}}$, is symmetric, that is $\Delta_a = \emptyset_e$, then the projector $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ is an orthogonal projector, i.e.

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta). \quad (3.10)$$

2. If the set $\Delta, \Delta \subseteq \sigma_{\mathcal{F}_{\mathbb{R}^+}}$, is not symmetric, i.e. $\Delta_a \neq \emptyset_e$, then the projector $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ is not orthogonal, i.e. $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) \neq \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}^*(\Delta)$.

Theorem 3.3. Let Δ_1 and Δ_2 are subsets of the spectrum $\sigma_{\mathcal{F}_{\mathbb{R}^+}}$ which satisfy the condition

$$\left(\Delta_1 \cup (-\Delta_1)\right) \cap \left(\Delta_2 \cup (-\Delta_2)\right) = \emptyset. \quad (3.11)$$

Then the image subspaces $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1)L^2(E)$ and $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_2)L^2(E)$ are mutually orthogonal, that is

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}^*(\Delta_2)\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_1) = 0. \quad (3.12)$$

In particular, if

$$\text{either } \Delta_1 \subset \sigma_{\mathcal{F}_{\mathbb{R}^+}}^+, \Delta_2 \subset \sigma_{\mathcal{F}_{\mathbb{R}^+}}^+, \text{ or } \Delta_1 \subset \sigma_{\mathcal{F}_{\mathbb{R}^+}}^-, \Delta_2 \subset \sigma_{\mathcal{F}_{\mathbb{R}^+}}^-, \quad (3.13a)$$

and moreover

$$\Delta_1 \cap \Delta_2 = \emptyset, \quad (3.13b)$$

then (3.12) holds.

By induction with respect to n , from (3.8) the following statement can be derived:

Proposition 3.1. Let $\Delta_k, \Delta_k \subset \sigma_{\mathcal{F}_{\mathbb{R}^+}}$, $1 \leq k \leq n$, be a finite sequence of sets possessing the properties:

- a). Each of the sets $\Delta_k, 1 \leq k \leq n$, is $\mathcal{F}_{\mathbb{R}^+}$ -admissible;
- b). The sets $\Delta_k, 1 \leq k \leq n$, are disjoint. This means that

$$\Delta_p \cap \Delta_q = \emptyset, \quad \forall p, q : 1 \leq p, q \leq n, p \neq q.$$

Then the set $\Delta = \bigcup_{1 \leq k \leq n} \Delta_k$ is $\mathcal{F}_{\mathbb{R}^+}$ -admissible, and

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) = \sum_{1 \leq k \leq n} \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_k). \quad (3.14)$$

The property of the mapping $\Delta \rightarrow \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ expressed as Proposition 3.1 can be naturally considered as *the additivity of this mapping with respect to Δ* .

In general, the mapping $\Delta \rightarrow \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ is not countably additive. If $\Delta_k, \Delta_k \subset \sigma_{\mathcal{F}_E}, 1 \leq k < \infty$, is a countable sequence of $\mathcal{F}_{\mathbb{R}^+}$ -admissible sets, then their union

$$\Delta = \bigcup_{1 \leq k < \infty} \Delta_k \quad (3.15)$$

may be a non-admissible set. Even if the set Δ , (3.15), is admissible and the sets Δ_k are pairwise disjoint, the equality

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) = \sum_{1 \leq k < \infty} \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_k).$$

may be violated. And what is more, it may happen that despite all the sets $\Delta_k, 1 \leq k < \infty$, and their union Δ are $\mathcal{F}_{\mathbb{R}^+}$ -admissible, the series in the right hand side of the last formula may diverge in any reasonable sense and even $\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_k)\| \rightarrow \infty$ as $k \rightarrow \infty$. The fact that the union Δ of the countable sequence of $\mathcal{F}_{\mathbb{R}^+}$ -admissible sets Δ_k is a \mathcal{F}_E -admissible set does not forbid the following pathology: *the property of the sets Δ_k be essentially separated from zero may be not uniform with respect to k* . Each of the sets Δ_k may be fully asymmetric: $\Delta_k = (\Delta_k)_a$, but their union Δ may be symmetric: $\Delta = \Delta_s$, hence $\mathcal{F}_{\mathbb{R}^+}$ -admissible, Lemma 3.1. However if the property of the sets Δ_k be essentially separated from zero is *not uniform with respect to k* , then the sequence $\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_k)\|, 1 \leq k < \infty$, is unbounded.

Nevertheless, some restricted property of countable additivity of the mapping $\Delta \rightarrow \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ takes place.

Theorem 3.4. *Let $\{\Delta_k\}_{1 \leq k < \infty}$ be a sequence of Borelian subsets of the spectrum $\sigma_{\mathcal{F}_{\mathbb{R}^+}}$ possessing the following properties:*

1. *The sets $\{\Delta_k\}_{1 \leq k < \infty}$ are pairwise disjoint:*

$$\Delta_p \cap \Delta_q = \emptyset \quad \forall p, q : 1 \leq p, q < \infty, p \neq q. \quad (3.16)$$

2. The sequence $\{(\Delta_k)_a\}_{1 \leq k < \infty}$ of the asymmetric parts $(\Delta_k)_a$ of the sets Δ_k is uniformly essentially separated from zero, that is

$$\inf_k \text{ess dist}((\Delta_k)_a, 0) > 0. \quad (3.17)$$

Then the set $\Delta = \bigcup_{1 \leq k < \infty} \Delta_k$ is $\mathcal{F}_{\mathbb{R}^+}$ -admissible and the equality

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) = \sum_{1 \leq k < \infty} \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_k). \quad (3.18)$$

holds, where the series in the right hand side of (3.18) converges strongly.

Lemma 3.3. Let the set Δ , $\Delta \subseteq \sigma(\mathcal{F}_{\mathbb{R}^+})$ be non-empty: $\Delta \neq \emptyset_e$.

1. If the set Δ is symmetric, i.e. $\text{mes } \Delta_a = 0$, then

$$\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\| = 1.$$

2. If the set is not symmetric, i.e. $\Delta_a \neq \emptyset_e$, then

$$\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\| = \frac{1}{2d} \sqrt{1 + 2d^2},$$

where $d = \text{ess dist}(\Delta_a, 0)$. In particular,

$$\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\| > 1.$$

Definition 3.5. To every $\mathcal{F}_{\mathbb{R}^+}$ -admissible set Δ , $\Delta \subseteq \sigma(\mathcal{F}_E)$, we relate the subspace

$$\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) \stackrel{\text{def}}{=} \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)L^2(\mathbb{R}^+), \quad (3.19)$$

which is the image of the projector $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}$.

Remark 3.1. For every admissible Δ , the subspace $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ is closed because it is the null-subspace of the bounded operator $\mathcal{I} - \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$.

Since $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\emptyset) = 0$ and $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+})) = \mathcal{I}$,

$$\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\emptyset) = 0, \quad \mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+})) = L^2(\mathbb{R}^+).$$

In view of (3.9), the subspace $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta)$ is the null-space of the projector $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$:

$$\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta) = \{x \in L^2(\mathbb{R}_+) : \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)x = 0\} \quad (3.20)$$

and the subspaces $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ and $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta)$ are complementary:

$$\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) \dot{+} \mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta) = L^2(\mathbb{R}_+). \quad (3.21)$$

(The sum in (3.21) is direct).

Since the projection operator $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ is a function of the operator $\mathcal{F}_{\mathbb{R}^+}$, it commutes with $\mathcal{F}_{\mathbb{R}^+}$:

$$\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\mathcal{F}_{\mathbb{R}^+} = \mathcal{F}_{\mathbb{R}^+}\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta). \quad (3.22)$$

From (3.22) it follows that the pair of complementary subspaces $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ and $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta)$, (3.21), reduces the operator $\mathcal{F}_{\mathbb{R}^+}$:

$$\begin{aligned} \mathcal{F}_{\mathbb{R}^+}\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta) &\subseteq \mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta), \\ \mathcal{F}_{\mathbb{R}^+}\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta) &\subseteq \mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus \Delta). \end{aligned} \quad (3.23)$$

In particular, the subspace $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ is invariant with respect to the operator $\mathcal{F}_{\mathbb{R}^+}$, and one can consider the restriction $\mathcal{F}_{\mathbb{R}^+}(\Delta)$ of the operator $\mathcal{F}_{\mathbb{R}^+}$ onto its invariant subspace $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$:

$$\mathcal{F}_{\mathbb{R}^+}(\Delta) \stackrel{\text{def}}{=} \mathcal{F}_{\mathbb{R}^+}|_{\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)}. \quad (3.24)$$

Theorem 3.5. *Let Δ be an admissible subset of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ of the operator $\mathcal{F}_{\mathbb{R}^+}$. The spectrum of the operator $\mathcal{F}_{\mathbb{R}^+}(\Delta)$, which acts in the space $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$, is the essential closure $\text{ess clos}(\Delta)$ of the set Δ :*

$$\sigma(\mathcal{F}_{\mathbb{R}^+}(\Delta)) = \text{ess clos}(\Delta). \quad (3.25)$$

Theorem 3.5 justifies the following

Definition 3.6. *Let $\Delta, \Delta \in \sigma(\mathcal{F}_{\mathbb{R}^+})$, be a $\mathcal{F}_{\mathbb{R}^+}$ -admissible set. The projector $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ defined by (3.3) is said to be the $\mathcal{F}_{\mathbb{R}^+}$ spectral projector corresponding to the set Δ .*

The subspace $\mathcal{H}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ – the image of the operator $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ – is said to be the $\mathcal{F}_{\mathbb{R}^+}$ spectral subspace corresponding to the set Δ .

Definition 3.7. *The operator-valued function $\Delta \rightarrow \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ which is defined on the set of all $\mathcal{F}_{\mathbb{R}^+}$ -admissible subsets Δ of the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ and whose values are spectral projectors $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ of the operator $\mathcal{F}_{\mathbb{R}^+}$ is said to be the spectral measure of the operator $\mathcal{F}_{\mathbb{R}^+}$.*

We recall that the spectral measure of $\mathcal{F}_{\mathbb{R}^+}$ possesses some property of sigma additivity. See Theorem 3.4.

For $0 < \varepsilon \leq 1/\sqrt{2}$, let

$$\Delta_+(\varepsilon) = e^{i\pi/4}[\varepsilon, 1/\sqrt{2}], \quad \Delta_-(\varepsilon) = e^{i\pi/4}[-1/\sqrt{2}, -\varepsilon]. \quad (3.26)$$

Each of the sets $\Delta_+(\varepsilon), \Delta_-(\varepsilon)$ with $\varepsilon > 0$ is $\mathcal{F}_{\mathbb{R}^+}$ -admissible, however the norms of spectral projectors $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_+(\varepsilon)), \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_-(\varepsilon))$ tend to ∞ as $\varepsilon \rightarrow +0$. Indeed, the sets $\Delta_{\pm}(\varepsilon)$ are fully asymmetric:

$$\Delta_+(\varepsilon) = (\Delta_+(\varepsilon))_a, \quad \Delta_-(\varepsilon) = (\Delta_+(\varepsilon))_a.$$

It is clear that

$$\text{ess dist}((\Delta_+(\varepsilon))_a, 0) = \varepsilon, \quad \text{ess dist}((\Delta_-(\varepsilon))_a, 0) = \varepsilon$$

According to Lemma 3.3,

$$\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_+(\varepsilon))\| = \frac{1}{2\varepsilon} \sqrt{1 + 2\varepsilon^2}, \quad \|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_-(\varepsilon))\| = \frac{1}{2\varepsilon} \sqrt{1 + 2\varepsilon^2}. \quad (3.27)$$

In particular,

$$\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_+(\varepsilon))\| \rightarrow +\infty, \quad \|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_-(\varepsilon))\| \rightarrow +\infty \quad \text{as } \varepsilon \rightarrow +0. \quad (3.28)$$

At the same time, the set

$$\Delta(\varepsilon) = \Delta_+(\varepsilon) \cup \Delta_-(\varepsilon) \quad (3.29)$$

is symmetric: $(\Delta(\varepsilon))_a = \emptyset$. According to Lemma 3.3,

$$\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta(\varepsilon))\| = 1 \quad \text{for every } \varepsilon > 0,$$

or

$$\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_+(\varepsilon)) + \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_-(\varepsilon))\| = 1 \quad \text{for every } \varepsilon > 0. \quad (3.30)$$

Sums of projectors from two unbounded families form a bounded family of projectors.

The family $\{\Delta(\varepsilon)\}_{\varepsilon>0}$ is monotonic: $\Delta(\varepsilon_1) \subseteq \Delta(\varepsilon_2)$ if $\varepsilon_1 > \varepsilon_2$. Moreover, $\bigcup_{\varepsilon>0} \Delta(\varepsilon) = \sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus 0$. Since $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus 0) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\sigma(\mathcal{F}_{\mathbb{R}^+})) = \mathcal{I}$, then, according to Theorem 3.4, the following assertion holds:

Lemma 3.4. *The estimate*

$$\|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta(\varepsilon))\| = 1 \quad \text{for every } \varepsilon > 0. \quad (3.31)$$

and the limiting relation

$$\lim_{\varepsilon \rightarrow +0} \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta(\varepsilon)) = \mathcal{I}, \quad (3.32)$$

hold, where convergence is the strong convergence of operators.

Corollary 3.1. *As we saw,*

$$\sup_{\Delta} \|\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\| = \infty, \quad (3.33)$$

where Δ runs over the class of all $\mathcal{F}_{\mathbb{R}^+}$ -admissible sets. From this it follows that the family $\{\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\}$ of spectral projectors is not similar to an orthogonal family of projectors.

Lemma 3.5. *By contrast with (3.33),*

$$\|\mathcal{F}_{\mathbb{R}^+} \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\| \leq \frac{\sqrt{2}+1}{2\sqrt{2}} \quad \text{for every admissible } \Delta. \quad (3.34)$$

In fact, following estimate holds.

Lemma 3.6. *Let $h(\zeta)$ be any $\mathcal{F}_{\mathbb{R}^+}$ -admissible complex-valued function. Then*

1.

$$\|\mathcal{F}_{\mathbb{R}^+} h(\mathcal{F}_{\mathbb{R}^+})\| \leq \sqrt{\frac{3}{2}} \cdot \operatorname{ess\,sup}_{\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})} |h(\zeta)|. \quad (3.35a)$$

2. *If moreover the function h is real-valued: $h(\zeta) \in \mathbb{R}$ for $\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})$, then*

$$\|\mathcal{F}_{\mathbb{R}^+} h(\mathcal{F}_{\mathbb{R}^+})\| \leq \operatorname{ess\,sup}_{\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})} |h(\zeta)|. \quad (3.35b)$$

3. *If the function h takes non-negative values: $h(\zeta) \in [0, \infty)$ for $\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})$, then*

$$\|\mathcal{F}_{\mathbb{R}^+} h(\mathcal{F}_{\mathbb{R}^+})\| \leq \frac{\sqrt{2}+1}{2\sqrt{2}} \operatorname{ess\,sup}_{\zeta \in \sigma(\mathcal{F}_{\mathbb{R}^+})} |h(\zeta)|. \quad (3.35c)$$

4. Functions of the operator $\mathcal{F}_{\mathbb{R}^+}$ as integrals over its spectral measure.

After the spectral projectors $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)$ were introduced, (3.3), and investigated, (see in particular Lemma (3.5)), the question arises: how to represent the original operator $\mathcal{F}_{\mathbb{R}^+}$ in terms of these spectral projectors. Our goal here is to give a meaning to the representation

$$\mathcal{F}_{\mathbb{R}^+} = \int_{\sigma_{\mathcal{F}_E}} \zeta \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta), \quad (4.1)$$

and more generally,

$$f(\mathcal{F}_{\mathbb{R}^+}) = \int_{\sigma_{\mathcal{F}_E}} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta). \quad (4.2)$$

We emphasize that the operator $\mathcal{F}_{\mathbb{R}^+}$ is non-normal, the family $\{\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta)\}$ is not orthogonal and even unbounded: (3.33). However it turns out that if the interval Δ is essentially separated from zero:

$$\text{ess dist}(\Delta, 0) > 0, \quad (4.3)$$

and the function $f(\zeta)$ is bounded on Δ , then the integral $\int_{\Delta} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$ can be provided with a meaning.

Namely, let $g(\zeta)$ be a simple function, that is the function of the form

$$g(\zeta) = \sum_k a_k \mathbb{1}_{\Delta_k}(\zeta), \quad (4.4)$$

where a_k are some complex numbers, and the collections Δ_k of sets forms a partition (finite) of the original set Δ : $\Delta = \bigcup_k \Delta_k$, $\Delta_p \cap \Delta_q = \emptyset$, $p \neq q$. We define the integral $\int_{\Delta} g(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$ as

$$\int_{\Delta} g(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) \stackrel{\text{def}}{=} \sum_k a_k \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_k). \quad (4.5)$$

The value of the sum in the right hand side of (4.5) does not depend on the representation of the function g in the form (4.4). So, the value

in the left hand side of (4.5) is well defined. From the other hand, decoding definition of $\mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_k)$ as $\mathbb{1}_{\Delta_k}(\mathcal{F}_{\mathbb{R}^+})$, we have

$$\sum_k a_k \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta_k) = \sum_k a_k \mathbb{1}_{\Delta_k}(\mathcal{F}_{\mathbb{R}^+}) = \left(\sum_k a_k \mathbb{1}_{\Delta_k} \right)(\mathcal{F}_{\mathbb{R}^+}) = g(\mathcal{F}_{\mathbb{R}^+}),$$

and finally,

$$\int_{\Delta} g(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) = g(\mathcal{F}_{\mathbb{R}^+}). \quad (4.6)$$

So for any simple function $g(\zeta)$ vanishing outside the set Δ , where Δ is separated from zero, the integral $\int_{\Delta} g(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$ is well defined and is interpreted as a function of the operator $\mathcal{F}_{\mathbb{R}^+}$ in the sense of the above introduced functional calculus.

Given a function f bounded on Δ and vanishing outside of Δ , there exists sequence f_n of simple functions vanishing outside of Δ which converges to f uniformly on Δ :

$$\overline{\lim}_{n \rightarrow \infty} \sup_{\zeta \in \Delta} |f(\zeta) - f_n(\zeta)| = 0.$$

The integral $\int_{\Delta} f(\zeta) \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ will be defined as the limit of integrals $\int_{\Delta} f_n(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$ of simple functions f_n if we justify that such a limit exists and does not depend on the approximating sequence $\{f_n\}$.

If $h(\zeta)$ be a function essentially bounded on Δ and vanishing outside of Δ , then

$$\|h(\mathcal{F})\| \leq (1 + 1/d) \sup_{\zeta \in \Delta} |h(\zeta)|,$$

where $d = \text{ess dist}(\Delta, 0)$. (See (2.24) and (2.17)). Applying this estimate to $h = f - f_n$, we see that $\|f(\mathcal{F}_{\mathbb{R}^+}) - f_n(\mathcal{F}_{\mathbb{R}^+})\| \rightarrow 0$ as $n \rightarrow \infty$. The convergence here is a convergence in the uniform operator topology. According to (4.6), this can be presented as

$$\left\| f(\mathcal{F}_{\mathbb{R}^+}) - \int_{\Delta} f_n(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) \right\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, there exists the limit of integrals $\int_{\Delta} f_n(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$. We declare this limit as $\int_{\Delta} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$:

$$\int_{\Delta} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \int_{\Delta} f_n(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta),$$

where convergence is the convergence in the norm of operators acting in $L^2(\mathbb{R}_+)$.

So the integral $\int_{\Delta} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$ is defined if Δ is any subset of $\sigma(\mathcal{F}_{\mathbb{R}^+})$ separated from zero and f is any function bounded on Δ and vanishing outside Δ . Moreover, this integral can be interpreted as a function f of the operator \mathcal{F}_E in the sense of Definition 2.6:

$$\int_{\Delta} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) = f(\mathcal{F}_{\mathbb{R}^+}).$$

Let now f be any bounded function defined on the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$. (We emphasize that the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ is not separated from zero, but contains the zero point, which is the singular point in some sense: see (3.28).) The integral $\int_{\sigma_{\mathcal{F}_E}} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$ will be defined as an improper integral. We remove a *symmetric* ε -neighborhood V_ε of zero

$$V_\varepsilon = (-\varepsilon e^{i\pi/4}, \varepsilon e^{i\pi/4}), \quad (4.7)$$

from the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$ and integrate f over the set $\Delta(\varepsilon) = \sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus V_\varepsilon$. (This is the same set $\Delta(\varepsilon)$ that was already defined in (3.26), (3.29).)

The set $\Delta(\varepsilon)$ is separated from zero, so the integral $\int_{\Delta(\varepsilon)} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta)$ is already defined. Then we pass to the limit as $\varepsilon \rightarrow +0$. If the limits exists in some sense, we declare the limiting operator as the integral

$$\int_{\sigma(\mathcal{F}_{\mathbb{R}^+})} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) x \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow +0} \int_{\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus V_\varepsilon} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) x. \quad (4.8)$$

Lemma 4.1. *We assume that f is a $\mathcal{F}_{\mathbb{R}^+}$ -admissible function.*

Then the limit in (4.8) exists in the sense of strong convergence, that is for every $x \in L^2(\mathbb{R}_+)$,

$$\left\| \int_{\sigma(\mathcal{F}_{\mathbb{R}^+})} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) x - \int_{\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus V_\varepsilon} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) x \right\|_{L^2(\mathbb{R}^+)} \rightarrow 0 \text{ as } \varepsilon \rightarrow +0. \quad (4.9)$$

Moreover

$$\int_{\sigma(\mathcal{F}_{\mathbb{R}^+})} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) = f(\mathcal{F}_{\mathbb{R}^+}), \quad (4.10)$$

where the operator $f(\mathcal{F}_{\mathbb{R}^+})$ is defined in the sense of Definition 2.6.

Proof. To justify the limiting relation (4.9) and to establish the equality (4.10), we observe that

$$\int_{\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus V_\varepsilon} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) = \int_{\Delta(\varepsilon)} \mathbb{1}_{\Delta(\varepsilon)}(\zeta) f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) = (\mathbb{1}_{\Delta(\varepsilon)} f)(\mathcal{F}_{\mathbb{R}^+}).$$

(For functions vanishing outside the set $\Delta(\varepsilon)$, which is separated from zero, the equality (4.10) is already established. In the present case, we apply the equality (4.10) to the function $\mathbb{1}_{\Delta(\varepsilon)}(\zeta)f(\zeta)$.) Since

$$(\mathbb{1}_{\Delta(\varepsilon)} f)(\mathcal{F}_{\mathbb{R}^+}) = \mathbb{1}_{\Delta(\varepsilon)}(\mathcal{F}_{\mathbb{R}^+}) f(\mathcal{F}_{\mathbb{R}^+}) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta(\varepsilon)) f(\mathcal{F}_{\mathbb{R}^+}),$$

we have

$$\int_{\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus V_\varepsilon} f(\zeta) \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(d\zeta) = \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}(\Delta(\varepsilon)) f(\mathcal{F}_{\mathbb{R}^+}).$$

According to Lemma 3.4,

$$\lim_{\varepsilon \rightarrow +0} \mathcal{P}_{\mathcal{F}_E}(\Delta(\varepsilon)) f(\mathcal{F}_{\mathbb{R}^+}) = f(\mathcal{F}_{\mathbb{R}^+}),$$

where convergence is the strong convergence of operators. Thus under the assumptions of Lemma, there exists the strong limit in (4.8) and the equality (4.10) holds. \square

Remark 4.1. For every fixed $\varepsilon > 0$, the spectral measure $\Delta \rightarrow \mathcal{P}_{\mathcal{F}_{\mathbb{R}^+}}$, restricted on $\Delta : \Delta \subseteq \sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus V_\varepsilon$, is sigma-additive. Because of this, it is possible to integrate an arbitrary bounded measurable function $h(\zeta)$ over $\sigma(\mathcal{F}_{\mathbb{R}^+}) \setminus V_\varepsilon$. However, the spectral measure is not sigma-additive and even unbounded on the family of all $\mathcal{F}_{\mathbb{R}^+}$ -admissible sets Δ . (See (3.28).) Therefore it is impossible to integrate an arbitrary bounded function $h(\zeta)$ over the whole $\sigma(\mathcal{F}_{\mathbb{R}^+})$. We have to restrict ourselves by bounded functions h which furthermore have a certain symmetry near the point $\zeta = 0$. Moreover we have to interpret the integral over $\sigma(\mathcal{F}_{\mathbb{R}^+})$ as an improper integral. (See (4.8).)

This reflects the fact that the point $\zeta = 0$ is a spectral singularity for $\mathcal{F}_{\mathbb{R}^+}$. (See Remark 2.2.)

5. The selfadjoint differential operator \mathcal{L} which commutes with the operator $\mathcal{F}_{\mathbb{R}+}$.

1. It is well known that the eigenfunctions of the Fourier operator \mathcal{F} are the Hermite functions $h_n(t)$:

$$\psi_n(t) = e^{\frac{t^2}{2}} \left(\frac{d}{dt} \right)^k e^{-t^2}, \quad (n = 0, 1, 2, \dots).$$

The equality

$$\mathcal{F}\psi_n = i^n \psi_n, \quad n \in \mathbb{N}, \quad (5.1)$$

can be checked by direct (but a little bit involved) calculation. The following facts helps both to guess the system $\{\psi_n\}_{n \in \mathbb{N}}$ and to check the equalities (5.1) in an organized way.

1. The selfadjoint differential operator $\mathcal{L}_{\mathcal{F}}$ generated by the formal differential operator $L_{\mathcal{F}} = -\frac{d^2}{dt^2} + t^2$ commutes with the operator \mathcal{F} . If $x(t)$ is a smooth function and $x(t)$, $x'(t)$, $x''(t)$ decay fast enough as $t \rightarrow \pm\infty$, then

$$\mathcal{F}\mathcal{L}_{\mathcal{F}}x = \mathcal{L}_{\mathcal{F}}\mathcal{F}x; \quad (5.2)$$

For arbitrary x from the domain of definition $\mathcal{D}_{\mathcal{L}_{\mathcal{F}}}$ of the operator $\mathcal{L}_{\mathcal{F}}$, the equality (5.2) can be justified by passing to the limit.

2. The functions ψ_n are eigenfunctions of the operator $\mathcal{L}_{\mathcal{F}}$:

$$\mathcal{L}_{\mathcal{F}}\psi_n(t) = \lambda_n \psi_n(t), \quad \lambda_n = 2n + 1, \quad n \in \mathbb{N}. \quad (5.3)$$

This differential operator has no other eigenfunctions: if

$$\mathcal{L}_{\mathcal{F}}x = \lambda x, \quad x \in L^2(\mathbb{R}), \quad x \neq 0, \quad (5.4)$$

then $\lambda = \lambda_n$ and x is proportional to ψ_n for some $n \in \mathbb{N}$.

The spectral analysis of the operator $-\frac{d^2}{dt^2} + t^2$, which is the energy operator of the quantum harmonic oscillator, was done in [Dir, Chapter VI, sec.34]. The operators $\mathfrak{a} = \frac{d}{dt} + t$, $\mathfrak{a}^\dagger = -\frac{d}{dt} + t$ are involved essentially in this spectral analysis, which is purely algebraic. The fact that the eigenvalues λ_n of the operator $\mathcal{L}_{\mathcal{F}}$ are *simple* (i.e. the appropriate eigenspaces are one-dimensional) is of crucial importance. Applying (5.2) to the function $x = \psi_n$, we see that the function $\mathcal{F}\psi_n$, as well as the function ψ_n , is an eigenfunction of \mathcal{F} corresponding to

the eigenvalue λ_n . However, the eigenspace of $\mathcal{L}_{\mathcal{F}}$ corresponding to the eigenvalue λ_n is one-dimensional and is generated by ψ_n . Therefore $\mathcal{F}\psi_n = \zeta_n\psi_n$ for some $\zeta_n \in \mathbb{C}$. Since $\mathcal{F}^4 = \mathcal{I}$, $\zeta_n^4 = 1$. So, ζ_n can take only one of four values $1, -1, i, -i$. More detail analysis shows that (5.1) holds.

Since the spectrum of $\mathcal{L}_{\mathcal{F}}$ is of multiplicity one, the operator \mathcal{F} , as well as any operator commuting with $\mathcal{L}_{\mathcal{F}}$, can be interpreted as a function of the operator $\mathcal{L}_{\mathcal{F}}$:

$$\mathcal{F} = h(\mathcal{L}_{\mathcal{F}}), \quad (5.5)$$

where

$$h(\lambda_n) = i^n, \quad n \in \mathbb{N}. \quad (5.6)$$

In particular, we may take

$$h(\mu) = e^{-i\pi/4} \cdot e^{i\mu\pi/4}, \quad \mu \in \mathbb{R}.$$

(Actually the operator $h(\mathcal{L}_{\mathcal{F}})$ depends only on values of the function h at the points λ_n , $n \in \mathbb{N}$.)

2. To extend this way of reasoning to the operator $\mathcal{F}_{\mathbb{R}^+}$, we have firstly to find the operator \mathcal{L} which commutes with $\mathcal{F}_{\mathbb{R}^+}$. It turns out that this is the differential operator \mathcal{L} generated by the formal differential operator L :

$$(Lx)(t) = -\frac{d}{dt} \left(t^2 \frac{dx(t)}{dt} \right) \quad (5.7)$$

This *formal* operator L generates the *minimal* operator \mathcal{L}_{\min} and *maximal* operator \mathcal{L}_{\max} . Namely, L describes how act the operators \mathcal{L}_{\min} and \mathcal{L}_{\max} on functions from the appropriate domain of definition.

Definition 5.1. *The set \mathcal{A} is the set of complex valued functions $x(t)$ defined on the open half-axis \mathbb{R}^+ and satisfying the following conditions:*

1. *The derivative $\frac{dx(t)}{dt}$ of the function $x(t)$ exists at every point t of the interval \mathbb{R}^+ ;*
2. *The function $\frac{dx(t)}{dt}$ is absolutely continuous on every compact subinterval of the interval \mathbb{R}^+ ;*

Definition 5.2. *The set $\mathring{\mathcal{A}}$ is the set of complex-valued functions $x(t)$ defined on \mathbb{R}^+ and satisfied the following conditions:*

1. *The function $x(t)$ belongs to the set \mathcal{A} defined above;*
2. *The support $\text{supp } x$ of the function $x(t)$ is a compact subset of the open half-axis \mathbb{R}^+ : $(\text{supp } x) \Subset \mathbb{R}^+$.*

Definition 5.3. *The differential operator \mathcal{L}_{\max} is defined as follows:*

1. *The domain of definition $\mathcal{D}_{\mathcal{L}_{\max}}$ of the operator \mathcal{L}_{\max} is:*

$$\mathcal{D}_{\mathcal{L}_{\max}} = \{x : x(t) \in L^2(\mathbb{R}^+) \cap \mathcal{A} \text{ and } (Lx)(t) \in L^2(\mathbb{R}^+)\}, \quad (5.8a)$$

where $(Lx)(t)$ is defined² by (5.7).

2. *The action of the operator \mathcal{L}_{\max} is:*

$$\text{For } x \in \mathcal{D}_{\mathcal{L}_{\max}}, \quad \mathcal{L}_{\max}x = Lx. \quad (5.8b)$$

The operator \mathcal{L}_{\max} is said to be the maximal differential operator generated by the formal differential expression L .

The minimal differential operator \mathcal{L}_{\min} is the restriction of the maximal differential operator \mathcal{L}_{\max} on the set of functions which in some sense vanish at the endpoint of the interval \mathbb{R}^+ . The precise definition is presented below.

Definition 5.4. *The operator \mathcal{L}_{\min} is the closure³ of the operator $\mathring{\mathcal{L}}$:*

$$\mathcal{L}_{\min} = \text{clos}(\mathring{\mathcal{L}}), \quad (5.9a)$$

where the operator $\mathring{\mathcal{L}}$ is the restriction of the operator \mathcal{L}_{\max} :

$$\mathring{\mathcal{L}} \subset \mathcal{L}_{\max}, \quad \mathring{\mathcal{L}} = \mathcal{L}_{\max}|_{\mathcal{D}_{\mathring{\mathcal{L}}}}, \quad \mathcal{D}_{\mathring{\mathcal{L}}} = \mathcal{D}_{\mathcal{L}_{\max}} \cap \mathring{\mathcal{A}}. \quad (5.9b)$$

By \langle, \rangle we denote the standard scalar product in $L^2(\mathbb{R}^+)$:

$$\text{For } u, v \in L^2(\mathbb{R}^+), \quad \langle u, v \rangle = \int_0^\infty u(t) \overline{v(t)} dt.$$

The properties of the operators \mathcal{L}_{\min} and \mathcal{L}_{\max} :

1. *The operator \mathcal{L}_{\min} is symmetric:*

$$\langle \mathcal{L}_{\min} x, y \rangle = \langle x, \mathcal{L}_{\min} y \rangle, \quad \forall x, y \in \mathcal{D}_{\mathcal{L}_{\min}}; \quad (5.10)$$

In other words, the operator \mathcal{L}_{\min} is contained in its adjoint:

$$\mathcal{L}_{\min} \subseteq (\mathcal{L}_{\min})^*;$$

²Since $x \in \mathcal{A}$, the expression $(Lx)(t)$ is well defined.

³Since the operator $\mathring{\mathcal{L}}$ is symmetric, it is closable.

2. The operators \mathcal{L}_{\min} and \mathcal{L}_{\max} are mutually adjoint:

$$(\mathring{\mathcal{L}})^* = (\mathcal{L}_{\min})^* = \mathcal{L}_{\max}, \quad (\mathcal{L}_{\max})^* = \mathcal{L}_{\min}; \quad (5.11)$$

The fact that $(\mathring{\mathcal{L}})^* = \mathcal{L}_{\max}$ is a very general fact related to ordinary differential operators, regular or singular, of finite or infinite interval.

Let us calculate the deficiency indices of the symmetric operator \mathcal{L}_{\min} . In view of (5.11), we have to investigate the dimension of the space of solutions of the equation $\mathcal{L}_{\max}x = \lambda x$ for λ from the upper half plane and for λ from the lower half plane. The equation $\mathcal{L}_{\max}x = \lambda x$ is the differential equation of the form

$$-\frac{d}{dt}\left(t^2\frac{dx(t)}{dt}\right) = \lambda x(t), \quad 0 < t < \infty. \quad (5.12)$$

We are interested in solutions of this equation which belong to $L^2(\mathbb{R}^+)$.

The equation (5.12) can be solved explicitly. Seeking its solution on the form $x(t) = t^a$, we come to the equation

$$a(a+1) + \lambda = 0.$$

The roots of this equation are

$$a_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}, \quad a_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}. \quad (5.13)$$

These roots are different if $\lambda \neq \frac{1}{4}$. Thus if $\lambda \neq \frac{1}{4}$, the general solution of the differential equation (5.12) is of the form

$$x(t) = c_1 t^{a_1} + c_2 t^{a_2}, \quad (5.14)$$

where c_1, c_2 are arbitrary constants. If $\lambda = \frac{1}{4}$, the general solution of (5.12) is of the form

$$x(t) = c_1 t^{-1/2} + c_2 t^{-1/2} \ln t. \quad (5.15)$$

However the function $x(t)$ of the form (5.14) (or (5.15)) belongs to $L^2((0, \infty))$ only if $x(t) \equiv 0$. Thus, the following result is proved

Lemma 5.1. *Whatever $\lambda \in \mathbb{C}$ is, the differential equation (5.12) has no solutions $x(t) \not\equiv 0$ belonging to $L^2(\mathbb{R}^+)$.*

In particular, taking $\lambda = i$ and $\lambda = -i$, we see that the deficiency indices n_+ and n_- of the symmetric operator \mathcal{L}_{\min} are equal to zero. Applying the von Neumann criterion of the selfadjointness, we obtain

Lemma 5.2. *The differential operator \mathcal{L}_{\min} is self-adjoint.*

In other words, we prove that $\mathcal{L}_{\min} = \mathcal{L}_{\max}$.

Notation 5.1. *From now till the end this paper we use the notation \mathcal{L} for the operator $\mathcal{L}_{\min} = \mathcal{L}_{\max}$:*

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}_{\min} = \mathcal{L}_{\max} \quad (5.16)$$

Since $\mathcal{L} = \mathcal{L}_{\min}$,

$$\mathcal{L} = \text{clos } \mathring{\mathcal{L}}. \quad (5.17)$$

Since $\mathcal{L} = \mathcal{L}_{\max}$,

$$\mathcal{D}_{\mathcal{L}} = \{x : x \in \mathcal{A} \cap L^2(\mathbb{R}^+), Lx \in L^2(\mathbb{R}^+)\}. \quad (5.18)$$

3. The following relationship between the operators \mathcal{L} and $\mathcal{F}_{\mathbb{R}^+}$ helps in the spectral analysis of the operator $\mathcal{F}_{\mathbb{R}^+}$ in much the same as the relationship (5.2) helps in the spectral analysis of the operator \mathcal{F} .

Theorem 5.1. *The (selfadjoint) operator \mathcal{L} commutes with the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$, as well as with the adjoint operator $\mathcal{F}_{\mathbb{R}^+}^*$:*

1. *If $x \in \mathcal{D}_{\mathcal{L}}$, then $\mathcal{F}_{\mathbb{R}^+} x \in \mathcal{D}_{\mathcal{L}}$, $\mathcal{F}_{\mathbb{R}^+}^* x \in \mathcal{D}_{\mathcal{L}}$.*
2.
$$\mathcal{F}_{\mathbb{R}^+} \mathcal{L} x = \mathcal{L} \mathcal{F}_{\mathbb{R}^+} x, \quad \mathcal{F}_{\mathbb{R}^+}^* \mathcal{L} x = \mathcal{L} \mathcal{F}_{\mathbb{R}^+}^* x, \quad \forall x \in \mathcal{D}_{\mathcal{L}}. \quad (5.19)$$
3. *If $\mathcal{E}(\Delta)$ is the spectral projector of the operator \mathcal{L} corresponding to a Borelian subset Δ of the real axis, then*

$$\mathcal{F}_{\mathbb{R}^+} \mathcal{E}(\Delta) = \mathcal{E}(\Delta) \mathcal{F}_{\mathbb{R}^+}, \quad \mathcal{F}_{\mathbb{R}^+}^* \mathcal{E}(\Delta) = \mathcal{E}(\Delta) \mathcal{F}_{\mathbb{R}^+}^* \quad \forall \Delta. \quad (5.20)$$

Proof.

1°. Let $x \in \mathcal{D}_{\mathcal{L}}$. Then the function $\mathcal{F}_{\mathbb{R}^+} x$ is the Fourier transform of a summable finite function, hence $\mathcal{F}_{\mathbb{R}^+} x \in \mathcal{A}$. Since $\mathcal{F}_{\mathbb{R}^+} L^2(\mathbb{R}^+) \subseteq L^2(\mathbb{R}^+)$, and $x \in L^2(\mathbb{R}^+)$, then $\mathcal{F}_{\mathbb{R}^+} x \in L^2(\mathbb{R}^+)$. Thus

$$\mathcal{F}_{\mathbb{R}^+} \mathcal{D}_{\mathcal{L}} \in \mathcal{A} \cap L^2(\mathbb{R}^+). \quad (5.21)$$

2°. Let as before $x \in \mathcal{D}_{\mathcal{L}}$. Integrating by parts twice, we obtain

$$\int_0^\infty \left(-\frac{d}{d\xi} \left(\xi^2 \frac{dx(\xi)}{d\xi} \right) \right) e^{it\xi} d\xi = it \int_0^\infty x(\xi) \left(\frac{d}{d\xi} \left(\xi^2 e^{it\xi} \right) \right) d\xi. \quad (5.22)$$

Since the support of the function $x(t)$ is a compact subset of the open interval \mathbb{R}^+ , the terms outside the integral vanish. Transforming the integral in the right hand side of (5.22) and denoting $y(t) = \int_0^\infty x(\xi)e^{it\xi} d(\xi)$, we obtain

$$\begin{aligned} -it \int_0^\infty x(\xi) \left(\frac{d}{d\xi} \left(\xi^2 e^{it\xi} \right) \right) d\xi &= -it \int_0^\infty x(\xi) (it\xi^2 + 2\xi) e^{it\xi} d\xi = \\ &= t^2 \int_0^\infty x(\xi) \xi^2 e^{it\xi} d\xi - 2it \int_0^\infty x(\xi) \xi e^{it\xi} d\xi = \\ &= -t^2 \frac{d^2 y(t)}{dt^2} - 2t \frac{dy(t)}{dt} = -\frac{d}{dt} \left(t^2 \frac{dy(t)}{dt} \right). \end{aligned} \quad (5.23)$$

The equality (5.23) means that

$$\mathcal{F}_{\mathbb{R}^+} Lx = L\mathcal{F}_{\mathbb{R}^+} x, \quad \forall x \in \mathcal{D}_{\mathcal{L}}. \quad (5.24)$$

Since $x \in \mathcal{D}_{\mathcal{L}}$, $Lx \in L^2(\mathbb{R})$. Therefore $\mathcal{F}_{\mathbb{R}^+} Lx \in L^2(\mathbb{R}^+)$. In view of (5.21) and (5.24), $L(\mathcal{F}_{\mathbb{R}^+} x) \in L^2(\mathbb{R}^+)$. Thus

$$\mathcal{F}_E \mathcal{D}_{\mathcal{L}} \subseteq \mathcal{D}_{\mathcal{L}}. \quad (5.25)$$

3°. Let $x \in \mathcal{D}_{\mathcal{L}}$ now. In view of (5.17), there exists a sequence $x_n \in \mathcal{D}_{\mathcal{L}}$ such that $x_n \rightarrow x$, $\mathcal{L}x_n \rightarrow \mathcal{L}x$ as $n \rightarrow \infty$. (The convergence is the strong convergence, that is the convergence in $L^2(\mathbb{R}^+)$.) According to (5.24), for every n the equality

$$\mathcal{F}_{\mathbb{R}^+} \mathcal{L}x_n = \mathcal{L}\mathcal{F}_{\mathbb{R}^+} x_n \quad (5.26)$$

holds. The operator \mathcal{F}_E is continuous. Therefore $\mathcal{F}_{\mathbb{R}^+} x_n \rightarrow \mathcal{F}_{\mathbb{R}^+} x$, and $\mathcal{F}_{\mathbb{R}^+} \mathcal{L}x_n \rightarrow \mathcal{F}_{\mathbb{R}^+} \mathcal{L}x$ as $n \rightarrow \infty$. Now from (5.26) it follows that there exists limit of the sequence $\mathcal{L}(\mathcal{F}_{\mathbb{R}^+} x_n)$. Since the operator \mathcal{L} is closed, then $\mathcal{F}_{\mathbb{R}^+} x \in \mathcal{D}_{\mathcal{L}}$ and $\mathcal{F}_{\mathbb{R}^+} \mathcal{L}x = \mathcal{L}\mathcal{F}_{\mathbb{R}^+} x$. The inclusion $\mathcal{F}_{\mathbb{R}^+}^* x \in \mathcal{D}_{\mathcal{L}}$ and the equality $\mathcal{F}_{\mathbb{R}^+}^* \mathcal{L}x = \mathcal{L}\mathcal{F}_{\mathbb{R}^+}^* x$ can be established analogously.

4°. Since the operator \mathcal{L} is selfadjoint, its spectrum is real. In particular, for every non-real number z , the operator $\mathcal{L} - z\mathcal{I}$ is invertible, and its inverse operator $(\mathcal{L} - z\mathcal{I})^{-1}$ is bounded and defined everywhere. Taking $x = (\mathcal{L} - z\mathcal{I})^{-1}y$ in (5.19), where y is an arbitrary vector, we obtain that

$$(\mathcal{L} - z\mathcal{I})^{-1} \mathcal{F}_{\mathbb{R}^+} = \mathcal{F}_{\mathbb{R}^+} (\mathcal{L} - z\mathcal{I})^{-1} \quad \forall z : \operatorname{Re} z \neq 0. \quad (5.27)$$

The equality (5.20) is a consequence of (5.27). \square

Theorem 5.1 suggest to interpret the operator $\mathcal{F}_{\mathbb{R}^+}$ as a function of the operator \mathcal{L} and to use the spectral analysis of the *self-adjoint* operator \mathcal{L} for study of the *non-normal* operator $\mathcal{F}_{\mathbb{R}^+}$. However since the spectrum of the operator \mathcal{L} is of *multiplicity two*, this function should be not a *scalar valued* function, but a *matrix-valued* one.

6. Spectral analysis of the operator \mathcal{L} . The functional model of the operator \mathcal{L} .

1°. The spectral analysis of the operator \mathcal{L} can be reduced to the spectral analysis of the operator $-\frac{d^2}{ds^2}$ in $L^2(\mathbb{R})$. Changing variables

$$t = e^s, \quad -\infty < s < \infty, \quad z(s) = e^{s/2}x(e^s), \quad (6.1)$$

we reduce the equation (5.12) to the form

$$-\frac{d^2 z(s)}{ds^2} + \frac{1}{4}z(s) = \lambda z(s), \quad -\infty < s < \infty. \quad (6.2)$$

The correspondence

$$z = Vx, \quad \text{where } z(s) = e^{s/2}x(e^s), \quad (6.3)$$

is an unitary operator from $L^2(\mathbb{R}^+, dt)$ onto $L^2(\mathbb{R}, ds)$:

$$\int_0^\infty |x(t)|^2 dt = \int_{-\infty}^\infty |z(s)|^2 ds. \quad (6.4)$$

The operator \mathcal{L} is unitarily equivalent to the operator $\mathcal{T} + \frac{1}{4}I$:

$$\mathcal{L} = V^{-1} \left(\mathcal{T} + \frac{1}{4}I \right) V, \quad (6.5)$$

where

$$(\mathcal{T}z)(s) = -\frac{d^2 z(s)}{ds^2} \quad (6.6)$$

is the differential operator in $L^2(\mathbb{R})$ defined on the "natural" domain. The spectral structure of the operator \mathcal{T} is well known. Its spectrum

$\sigma(\mathcal{T})$ is absolutely continuous of *multiplicity two* and fills the positive half-axis: $\sigma(\mathcal{T}) = [0, \infty)$. The (generalized) eigenfunctions of the operator \mathcal{T} corresponding to the point $\rho \in (0, \infty)$ are

$$z_+(s, \mu) = e^{i\mu s}, \quad z_-(s, \mu) = e^{-i\mu s}, \quad s \in \mathbb{R}, \quad (6.7)$$

where $\mu = \rho^{1/2} > 0$. Changing variable in the expressions (6.7) for eigenfunctions of the operator \mathcal{T} according to (6.1), we come to the functions

$$\psi^+(t) = t^{-\frac{1}{2}+i\mu}, \quad \psi^-(t) = t^{-\frac{1}{2}-i\mu}, \quad t \in \mathbb{R}^+, \quad \mu \in \mathbb{R}^+. \quad (6.8)$$

Both of the functions $\psi^+(t, \mu), \psi^-(t, \mu)$ are eigenfunctions of the operator \mathcal{L} corresponding to *the same* eigenvalue $\lambda(\mu)$,

$$\lambda(\mu) = \mu^2 + 1/4, \quad \mu \in \mathbb{R}^+. \quad (6.9)$$

$$\mathcal{L}\psi^+(t, \mu) = \lambda(\mu)\psi^+(t, \mu), \quad \mathcal{L}\psi^-(t, \mu) = \lambda(\mu)\psi^-(t, \mu). \quad (6.10)$$

In view of (6.5), the spectral properties of the operator \mathcal{T} can be reformulated as the spectral properties of the operator \mathcal{L} . Reindexing the spectral parameter μ in such a manner that the value of the parameter to be coincide with the eigenvalue, we come to the functions

$$\varphi^+(t, \lambda) = \psi^+(t, \mu(\lambda)), \quad \varphi^-(t, \lambda) = \psi^-(t, \mu(\lambda)). \quad (6.11)$$

where

$$\mu = \mu(\lambda) = \sqrt{\lambda - \frac{1}{4}}, \quad \mu > 0, \quad 1/4 < \lambda < \infty. \quad (6.12)$$

$$\mathcal{L}\varphi^+(t, \lambda) = \lambda\varphi^+(t, \lambda), \quad \mathcal{L}\varphi^-(t, \lambda) = \lambda\varphi^-(t, \lambda), \quad 1/4 < \lambda < \infty. \quad (6.13)$$

In what follow we work mainly with the system

$$\{\psi^+(t, \mu), \psi^-(t, \mu)\}_{\mu \in (0, \infty)}$$

of "non-reindexed" eigenfunctions, but not with the system

$$\{\varphi^+(t, \lambda), \varphi^-(t, \lambda)\}_{\lambda \in (1/4, \infty)}$$

of "reindexed" eigenfunctions. The reindexing procedure is useful if we would like to feed the eigenfunctions to the operator \mathcal{L} in a most natural way. However, the operator \mathcal{L} plays the heuristic role only. What we actually need these are eigenfunctions of \mathcal{L} but not \mathcal{L} itself.

The spectrum $\sigma(\mathcal{L})$ of the operator \mathcal{L} is absolutely continuous of multiplicity two and fills of the semi-infinite interval:

$$\sigma(\mathcal{L}) = [1/4, \infty). \quad (6.14)$$

To the point $\lambda \in (\frac{1}{4}, \infty)$ of the spectrum of the operator \mathcal{L} corresponds the *two-dimensional* "generalized eigenspace" generated by the 'generalized' eigenfunctions $\psi^+(t, \mu(\lambda))$, $\psi^-(t, \mu(\lambda))$, $\mu(\lambda)$ is defined in (6.12).

Given $\mu \in (0, \infty)$, the "eigenfunctions" $\psi^+(t, \mu)$, $\psi^-(t, \mu)$ do not belong to the space $L^2(\mathbb{R}^+, dt)$, but almost belong. Their averages with respect to the spectral parameter

$$\frac{1}{2\varepsilon} \int_{\mu-\varepsilon}^{\mu+\varepsilon} \psi^\pm(t, \zeta) d\zeta = t^{-\frac{1}{2} \pm i\mu} \frac{\sin(\varepsilon \ln t)}{\varepsilon \ln t}$$

over an arbitrary small interval $(\mu - \varepsilon, \mu + \varepsilon) \subset (0, \infty)$ already belongs to $L^2((0, \infty), dt)$. These eigenfunctions satisfy the generalized "orthogonality relations":

$$\begin{aligned} \int_0^\infty \overline{\psi^-(t, \mu_2)} \psi^+(t, \mu_1) dt &= 0, \\ \int_0^\infty \overline{\psi^+(t, \mu_2)} \psi^+(t, \mu_1) dt &= 2\pi \delta(\mu_1 - \mu_2), \\ \int_0^\infty \overline{\psi^-(t, \mu_2)} \psi^-(t, \mu_1) dt &= 2\pi \delta(\mu_1 - \mu_2), \end{aligned}$$

$\forall \mu_1, \mu_2 > 0$, where δ is the Dirac δ -function. (6.15)

The integrals in (6.15) diverge, so the relations (6.15) are nonsense if they are understood literally. Nevertheless the equalities (6.15) can be provide with a meaning in the sense of distributions.

However we prefer to stay on the 'classical' point of view, and to to formulate the 'orthogonality properties' of the 'eigenfunctions' $\psi_\pm(t, \lambda)$ in the language of the L^2 -theory of the Fourier integrals.

Notation. In what follows we use the matrix notation, matrix language, and matrix operations. We organize the pair $\psi_+(t, \mu)$, $\psi_-(t, \mu)$,

(6.8), into the matrix-row

$$\psi(t, \mu) = \begin{bmatrix} \psi_+(t, \mu) & \psi_-(t, \mu) \end{bmatrix}. \quad (6.16a)$$

According to the matrix algebra notation, the matrix adjoint to the matrix-column $\psi(t, \mu)$ is the matrix-column

$$\psi^*(t, \mu) = \begin{bmatrix} \overline{\psi_+(t, \mu)} \\ \overline{\psi_-(t, \mu)} \end{bmatrix}. \quad (6.16b)$$

In this notation, the orthogonality relation (6.15) can be presented as

$$\int_{t \in \mathbb{R}^+} \psi^*(t, \mu_2) \psi(t, \mu_1) dt = 2\pi \delta(\mu_1 - \mu_2) I, \quad (6.17)$$

where I is the 2×2 identity matrix.

2°. One of fundamental results of the theory of selfadjoint operators can be roughly formulated as follows.

Every selfadjoint operator is unitary equivalent to a multiplication operator in a space of square integrable vector functions on a subset of the real axis.

In the case of the operator \mathcal{L} , this is a space of two-component vector functions defined on the positive half-axis \mathbb{R}^+ which are square integrable with respect to the Lebesgue measure on \mathbb{R}^+ . (The number *two* is the spectral multiplicity of the operator \mathcal{L} .) The precise definition is as follows.

Definition 6.1. *The space \mathcal{K} is the space of all two component vector functions $y(\mu) = \begin{bmatrix} y^+(\mu) \\ y^-(\mu) \end{bmatrix}$ which are defined on the positive half-axis $\mu \in \mathbb{R}^+$, are Borel measurable and satisfy the condition*

$$\int_{\mu \in \mathbb{R}^+} y^*(\mu) y(\mu) \frac{d\mu}{2\pi} < \infty. \quad (6.18)$$

The set \mathcal{K} provided by pointwise algebraic operations and the scalar product

$$\langle y_1, y_2 \rangle_{\mathcal{K}} = \int_{\mu \in \mathbb{R}^+} y_1^*(\mu) y_2(\mu) \frac{d\mu}{2\pi}. \quad (6.19)$$

is a Hilbert space. (Strictly speaking elements of \mathcal{K} are equivalency classes of vector functions.)

We use the terminology the model space for the Hilbert space \mathcal{K} .

3°. With the system $\{\psi(t, \mu)\}_{\mu \in \mathbb{R}^+}$ of eigenfunctions of the operator \mathcal{L} we relate the ‘Fourier transform’ $x(t) \rightarrow \hat{x}(\mu)$ in this eigenfunctions. For a function $x \in L^2(\mathbb{R}^+)$ compactly supported on \mathbb{R}^+ : $\text{supp } x \subseteq \mathbb{R}^+$, we set

$$\hat{x}(\mu) = \int_{t \in \mathbb{R}^+} \psi^*(t, \mu) x(t) dt, \quad \mu \in \mathbb{R}^+. \quad (6.20)$$

For a vector-function $y(\mu) \in \mathcal{K}$ compactly supported on \mathbb{R}^+ : $\text{supp } y \subseteq \mathbb{R}^+$, we set

$$\check{y}(t) = \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) y(\mu) \frac{d\mu}{2\pi}, \quad t \in \mathbb{R}^+. \quad (6.21)$$

Since $\psi(t, \mu)\psi^*(t, \mu) = 2t^{-1}$ for every $t \in \mathbb{R}^+, \mu \in \mathbb{R}^+$, the integrals in (6.20), (6.21) converge absolutely for every $\mu \in \mathbb{R}^+, t \in \mathbb{R}^+$ respectively. Thus for compactly supported $x \in L^2(\mathbb{R}^+)$, $y \in \mathcal{K}$, the functions $\hat{x}(\mu), \check{y}(t)$ are well defined function on the half-axis $\mu \in \mathbb{R}^+, t \in \mathbb{R}^+$ respectively.

Remark 6.1. *The transformations (6.20) and (6.21) are closely related to the Mellin transforms (direct and inverse). Let $x(t)$ be a function defined on \mathbb{R}^+ . The Mellin transform $(\mathcal{M}x)(\zeta)$ of the function x is defined as*

$$\hat{x}(\zeta) = (\mathcal{M}x)(\zeta) = \int_0^\infty t^\zeta x(t) \frac{dt}{t} \quad (6.22)$$

The inverse Mellin transform is

$$x(t) = (\mathcal{M}^{-1}x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-\zeta} \hat{x}(\zeta) d\zeta. \quad (6.23)$$

which is defined for those complex ζ for which the integral exists. The restriction of the Mellin transform (6.22) on the vertical line $\zeta = 1/2 + i\mu, \mu \in \mathbb{R}$, coincides essentially with the transform $x(t) \rightarrow \hat{x}(\mu)$ defined by (6.20). The inverse Mellin transform (6.23), with the value $c = 1/2$, coincides essentially with the transform $y(\mu) \rightarrow \check{y}(t)$ defined by (6.21).

Lemma 6.1.

1. Assume that $x \in L^2(\mathbb{R}^+)$ and $\text{supp } x \Subset \mathbb{R}^+$. Let us define the function $\hat{x}(\mu)$ by (6.20). Then the Parseval equality

$$\int_{\mu \in \mathbb{R}^+} \hat{x}^*(\mu) \hat{x}(\mu) \frac{d\mu}{2\pi} = \int_{t \in \mathbb{R}^+} \overline{x}(t) x(t) dt. \quad (6.24)$$

holds.

2. Assume that $y \in \mathcal{K}$ and $\text{supp } y \Subset \mathbb{R}^+$. Let us define the function $\check{y}(t)$ by (6.21). Then the Parseval equality

$$\int_{t \in \mathbb{R}^+} \overline{\check{y}}(t) \check{y}(t) dt = \int_{\mu \in \mathbb{R}^+} y^*(\mu) y(\mu) \frac{d\mu}{2\pi}. \quad (6.25)$$

holds.

This lemma says that the mappings $x(t) \rightarrow \hat{x}(\mu)$, $y(\mu) \rightarrow \check{y}(t)$, defined so far only for compactly supported $x \in L^2(\mathbb{R}^+)$, $y \in \mathcal{K}$, are isometric mappings from $L^2(\mathbb{R}^+)$ into \mathcal{K} and from \mathcal{K} into $L^2(\mathbb{R}^+)$ respectively. In particular, each of these two mappings is *uniformly continuous* on its domain of definition. The sets of compactly supported x, y are *dense subsets* of the spaces $L^2(\mathbb{R}^+)$ and \mathcal{K} respectively. Therefore the mapping $x(t) \rightarrow \hat{x}(\mu)$ can be extended to a continuous mapping from $L^2(\mathbb{R}^+)$ to \mathcal{K} defined on the whole space $L^2(\mathbb{R}^+)$. Analogously the mapping $y(\mu) \rightarrow \check{y}(t)$ can be extended to a continuous mapping from $L^2(\mathbb{R}^+)$ to \mathcal{K} defined on the whole space \mathcal{K} .

Definition 6.2.

1. The mapping $U, U : L^2(\mathbb{R}^+) \rightarrow \mathcal{K}$, is the continuous mapping defined on the whole space $L^2(\mathbb{R}^+)$ which acts as

$$(Ux)(\mu) = \int_{t \in \mathbb{R}^+} \psi^*(t, \mu) x(t) dt \quad (6.26)$$

for $x \in L^2(\mathbb{R})$ which are compactly supported on \mathbb{R}^+ .

2. The mapping $U^{-1}, U^{-1} : \mathcal{K} \rightarrow L^2(\mathbb{R}^+)$, is the continuous mapping defined on the whole space \mathcal{K} which acts as

$$(U^{-1}y)(t) = \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) y(\mu) \frac{d\mu}{2\pi} \quad (6.27)$$

for $y \in \mathcal{K}$ which are compactly supported on \mathbb{R}^+ .

(Here U^{-1} is a notation only. We do not claim for the present that the operator U^{-1} is the operator inverse to U .)

Remark 6.2. If $x \in L^2(\mathbb{R}^+)$, but x is not compactly supported on \mathbb{R}^+ , then the function $\psi(t, \mu)x(t)$ may be not integrable, i.e. the integral in (6.20) may not exist in the proper sense. In this case, the function Ux is defined by means of the above mentioned extension procedure rather by the integral (6.20). Nevertheless we will use the expression (6.20) for the function $\hat{x} = Ux$ as a notation no matter whether the integral in (6.20) exists or not as a Lebesgue integral. The same is also related to the function $\tilde{y} = U^{-1}y$.

Theorem 6.1.

1. The mapping U introduced in Definition 6.2 is an unitary mapping from the space $L^2(\mathbb{R}^+)$ onto the model space \mathcal{K} .
2. The mapping U^{-1} introduced in Definition 6.2 is an unitary mapping from the space \mathcal{K} onto the model space $L^2(\mathbb{R}^+)$.
3. The mappings U and U^{-1} are mutually inverse:

$$U^{-1}U = \mathbb{J}_{L^2(\mathbb{R}^+)}, \quad UU^{-1} = \mathbb{J}_{\mathcal{K}}. \quad (6.28)$$

Statement 3 of Theorem 6.1 claims that for arbitrary $x \in L^2(\mathbb{R}^+)$ the pair of formulas

$$\hat{x}(\mu) = \int_{t \in \mathbb{R}^+} \psi^*(t, \mu) x(t) dt, \quad (6.29a)$$

$$x(t) = \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) \hat{x}(\mu) \frac{d\mu}{2\pi} \quad (6.29b)$$

holds. This pair of formulas can be considered as the *expansion of arbitrary $x \in L^2(\mathbb{R}^+)$ in eigenfunction of the operator \mathcal{L} .*

Proof of Lemma 6.1 and Theorem 6.1.

Lemma 6.1 and Theorem 6.1 are paraphrases of the classical facts from the L^2 -theory of the Fourier integral. Given the function $z(s) \in L^2(\mathbb{R}, ds)$, its Fourier transform $\tilde{z}(\mu)$ is

$$\tilde{z}(\mu) = \int_{s \in \mathbb{R}} z(s) e^{-i\mu s} ds, \quad \mu \in \mathbb{R}.$$

We split the function $\tilde{z}(\mu)$ into the pair $\tilde{z}_+(\mu)$ and $\tilde{z}_-(\mu)$, both functions $\tilde{z}_+(\mu)$ and $\tilde{z}_-(\mu)$ are defined for $\mu \in \mathbb{R}^+$:

$$\tilde{z}_+(\mu) = \int_{s \in \mathbb{R}} z(s) e^{-i\mu s} ds, \quad \tilde{z}_-(\mu) = \int_{s \in \mathbb{R}} z(s) e^{i\mu s} ds, \quad \mu \in \mathbb{R}^+. \quad (6.30)$$

The Parseval identity and the inversion formula can be presented in the form

$$\int_{s \in \mathbb{R}} |z(s)|^2 ds = \int_{\mu \in \mathbb{R}^+} |\tilde{z}_+(\mu)|^2 \frac{d\mu}{2\pi} + \int_{\mu \in \mathbb{R}^+} |\tilde{z}_-(\mu)|^2 \frac{d\mu}{2\pi}, \quad (6.31)$$

and

$$z(s) = \int_{\mathbb{R}^+} \tilde{z}_+(\mu) e^{i\mu s} \frac{d\mu}{2\pi} + \int_{\mu \in \mathbb{R}^+} \tilde{z}_-(\mu) e^{-i\mu s} \frac{d\mu}{2\pi}. \quad (6.32)$$

Changing variable

$$x(t) = t^{-1/2} z(\ln t), \quad \mu(\lambda) = \sqrt{\lambda - 1/4},$$

(see (6.1)), we present the formulas (6.30) and (6.31) in the form (6.20) and (6.24) respectively. The inversion formula (6.32) corresponds to the formula (6.21), where $\hat{x}(\mu)$ from (6.20) is taken for $y(\mu)$ and $\check{y}(t) = x(t)$. This means that the operators U, U^{-1} are mutually inverse. \square

4°. The operator U , which is constructed from eigenfunctions of the operator \mathcal{L} , diagonalizes the operator \mathcal{L} . More precisely, this means that the operator $U\mathcal{L}U^{-1}$ is a ‘diagonal’ operator in the space \mathcal{K} .

To explain this, let us introduce the multiplication operator \mathcal{M} acting in the space \mathcal{K} .

Definition 6.3.

1. The domain of definition $\mathcal{D}_{\mathcal{M}}$ of the operator \mathcal{M} is

$$\mathcal{D}_{\mathcal{M}} = \{y \in \mathcal{K} : \lambda(\mu)y(\mu) \in \mathcal{K}\}, \quad (6.33)$$

where $\lambda(\mu)$ is defined in (6.9).

2. For $y \in \mathcal{D}_{\mathcal{M}}$,

$$(\mathcal{M}y)(\mu) = \lambda(\mu)y(\mu), \quad \mu \in \mathbb{R}^+. \quad (6.34)$$

Theorem 6.2. *The equality*

$$\mathcal{L} = U^{-1}\mathcal{M}U \quad (6.35)$$

holds. In particular,

$$U\mathcal{D}_{\mathcal{L}} = \mathcal{D}_{\mathcal{M}}.$$

Theorem 6.2 is a paraphrase of a standard result from the theory of Fourier integral. This result tell how to express the Fourier transform of the derivative of some function in terms of the Fourier transform of the function itself. We do not present the detail explanation. This theorem plays an heuristic role only. The only what we need is the expression (6.8) for generalized eigenfunction of the operator \mathcal{L} corresponding to the point $\mu = \mu(\lambda)$ of the spectrum \mathcal{L} .

Theorem 6.2 says that the differential operator \mathcal{L} is unitary equivalent to the multiplication operator \mathcal{M} . The operator \mathcal{M} may be considered as a *model* of the operator \mathcal{L} . Spectral properties of the operator \mathcal{L} can be reformulated in terms of spectral properties of the model operator \mathcal{M} . From the other hand, since the model operator is diagonal, to study spectral properties of the model operator is easier than to study spectral properties of the original operator \mathcal{L} .

7. Matrix-functional calculus for the operator \mathcal{L} .

In Theorem 6.2 it was stated that the differential operator \mathcal{L} is unitary equivalent to the multiplication operator \mathcal{M} in the space \mathcal{K} of vector-functions. This allows us to built functional calculus for the operator \mathcal{L} .

If $g(\mu)$, $f : \mathbb{R}^+ \rightarrow \mathbb{C}$, is a bounded complex-values measurable function, we define the operator \mathcal{M}_g , which acts in the model space \mathcal{K} , $\mathcal{M}_g : \mathcal{K} \rightarrow \mathcal{K}$, as

$$(\mathcal{M}_g y)(\mu) \stackrel{\text{def}}{=} g(\mu)y(\mu), \quad y \in \mathcal{K}. \quad (7.1)$$

In this notation, the equality (6.35) can be presented in the form

$$\mathcal{L} = U^{-1}\mathcal{M}_{\lambda}U, \quad (7.2)$$

where U , U^{-1} are the unitary operators defined in (6.26), (6.27), and the function $\lambda(\mu)$ is defined in (6.9).

This motivates the following definition.

If $h(\lambda)$ is a bounded measurable function defined on the spectrum $\sigma(\mathcal{L})$ of \mathcal{L} , (6.14), then the operator $h(\mathcal{L})$, $h(\mathcal{L}) : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$, is defined as

$$h(\mathcal{L}) \stackrel{\text{def}}{=} U^{-1} \mathcal{M}_g U. \quad (7.3)$$

where

$$g(\mu) = h(\lambda(\mu)), \quad (7.4)$$

$\lambda(\mu)$ is defined in (6.9).

The operator \mathcal{M}_g acts in the space \mathcal{K} of two-component *vector* functions and multiplies a vector-function $y(\mu)$ on the *scalar* function $g(\mu)$. However, it is natural to consider an operator which multiplies a vector-function $y(\mu)$ on a 2×2 *matrix*-valued function $G(\mu)$.

Definition 7.1.

1. $L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$ is the set of all 2×2 matrix functions which entries are complex valued functions defined almost everywhere on \mathbb{R}^+ and essentially bounded there.
2. The set $L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$ is provided by pointwise algebraic operation: the addition, the multiplication and the multiplication with complex scalars.
3. For $G(\mu) \in L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$, the norm $\|G\|_{L_{\mathfrak{M}_2}^\infty}$ is defined as

$$\|G\|_{L_{\mathfrak{M}_2}^\infty} \stackrel{\text{def}}{=} \text{ess sup}_{\mu \in \mathbb{R}^+} \|G(\mu)\|, \quad (7.5)$$

where for each μ , the expression $\|G(\mu)\|$ means the norm of the matrix $F(\mu)$ considered as an operator in the two-dimensional complex Euclidean space, and ess sup is the essential supremum with respect to the Lebesgue measure on \mathbb{R}^+ .

4. For $G(\mu) \in L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$, the conjugate matrix function $G^*(\mu)$ is defined as

$$G^*(\mu) \stackrel{\text{def}}{=} (G(\mu))^*, \quad \mu \in \mathbb{R}^+, \quad (7.6)$$

where where for each μ , the expression $G(\mu)^*$ means the matrix Hermitian conjugated to the matrix $G(\mu)$.

The set $L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$ provided by pointwise algebraic operations and the norm (7.5) is a Banach algebra.

Definition 7.2. Let $G(\mu)$ be a matrix function from $L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$.

The operator \mathcal{M}_G , which acts in the model space \mathcal{K} , $\mathcal{M}_G : \mathcal{K} \rightarrow \mathcal{K}$, is defined as

$$(\mathcal{M}_G y)(\mu) \stackrel{\text{def}}{=} G(\mu)y(\mu), \quad y \in \mathcal{K}. \quad (7.7)$$

Definition 7.3. Let $H(\lambda)$,

$$H(\lambda) = \begin{bmatrix} h_{++}(\lambda) & h_{+-}(\lambda) \\ h_{-+}(\lambda) & h_{--}(\lambda) \end{bmatrix}, \quad \lambda \in \sigma(\mathcal{L}), \quad (7.8)$$

be a measurable 2×2 matrix function with complex-valued entries $h_{\pm}(\lambda)$ defined almost everywhere on the spectrum $\sigma(\mathcal{L})$ of the operator \mathcal{L} . Assume that the matrix function H is essentially bounded on $\sigma(\mathcal{L})$:

$$\text{ess sup}_{\lambda \in \sigma(\mathcal{L})} \|H(\lambda)\| < \infty. \quad (7.9)$$

The operator $H(\mathcal{L})$, $H(\mathcal{L}) : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$, is defined as

$$H(\mathcal{L}) \stackrel{\text{def}}{=} U^{-1} \mathcal{M}_F U, \quad (7.10)$$

where U, U^{-1} are the unitary operators defined in (6.26), (6.27), and

$$F(\mu) = H(\lambda(\mu)), \quad \lambda(\mu) \text{ is defined in (6.9)}. \quad (7.11)$$

Lemma 7.1. The mapping $G \rightarrow \mathcal{M}_G$, is a norm preserving homomorphism of the algebra $L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$ of matrix functions into the algebra of all bounded operators in the model space \mathcal{K} :

1. If $G(\mu) \equiv I$, then $\mathcal{M}_G = \mathcal{I}_{\mathcal{K}}$, where I is the identity 2×2 matrix, and $\mathcal{I}_{\mathcal{K}}$ is the identity operator in \mathcal{K} .
2. If $G(\mu) = \alpha_1 G_1(\mu) + \alpha_2 G_2(\mu)$, where $G_1, G_2 \in L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, then $\mathcal{M}_G = \alpha_1 \mathcal{M}_{G_1} + \alpha_2 \mathcal{M}_{G_2}$.
3. If $G(\mu) = G_1(\mu) \cdot G_2(\mu)$, where $G_1, G_2 \in L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$, then $\mathcal{M}_G = \mathcal{M}_{G_1} \cdot \mathcal{M}_{G_2}$.
4. If $G \in L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$, then

$$(\mathcal{M}_G)^* = \mathcal{M}_{G^*}, \quad (7.12)$$

where G^* is the matrix function conjugated to the matrix-function G and \mathcal{M}_{G^*} is the operator conjugated to the operator \mathcal{M}_G with respect to the scalar product in \mathcal{K} .

5. If $G \in L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$, then

$$\|\mathcal{M}_G\|_{\mathcal{K} \rightarrow \mathcal{K}} = \|G\|_{L_{\mathfrak{M}_2}^\infty}. \quad (7.13)$$

Lemma 7.2. Let $\{G_n\}_{1 \leq n < \infty}$ be a sequence of 2×2 matrix functions, $G_n \in L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$ for every n .

We assume that

1. The sequence $\{G_n\}_{1 \leq n < \infty}$ is uniformly bounded, that is

$$\sup_n \|G_n\|_{L_{\mathfrak{M}_2}^\infty} < \infty. \quad (7.14)$$

2. For almost every $\mu \in \mathbb{R}^+$ there exists the limit of matrices $G_n(\mu)$:

$$\lim_{n \rightarrow \infty} G_n(\mu) = G(\mu). \quad (7.15)$$

Then the sequence of operators $\{\mathcal{M}_{G_n}\}_{1 \leq n < \infty}$ converges **strongly** to the operator \mathcal{M}_G :

$$\lim_{n \rightarrow \infty} \|\mathcal{M}_{G_n} y - \mathcal{M}_G y\|_{\mathcal{K}} = 0 \quad \text{for every } y \in \mathcal{K}, \quad (7.16)$$

Proof. According to (7.14), there exists a constant $C < \infty$ such that the norms of the matrices $G_n(\mu)$, $G(\mu)$ admit the estimates $\|G_n(\mu)\| \leq C$, $\|G(\mu)\| \leq C$ for almost every $\mu \in \mathbb{R}^+$. Therefore for every $y \in \mathcal{K}$, the inequalities

$$\|G_n(\mu)y(\mu) - G(\mu)y(\mu)\|_{\mathbb{C}^2}^2 \leq 4C^2 \|y(\mu)\|_{\mathbb{C}^2}^2 \quad (7.17)$$

hold for almost every $\mu \in \mathbb{R}^+$ and for every $n = 1, 2, 3, \dots$. From the condition (7.15) it follows that

$$\lim_{n \rightarrow \infty} \|G_n(\mu)y(\mu) - G(\mu)y(\mu)\|_{\mathbb{C}^2}^2 = 0 \quad \text{for almost every } \mu \in \mathbb{R}^+. \quad (7.18)$$

Since $y \in \mathcal{K}$,

$$\int_{\mu \in \mathbb{R}^+} \|y(\mu)\|_{\mathbb{C}^2}^2 \frac{d\mu}{2\pi} = \|y\|_{\mathcal{K}}^2 < \infty. \quad (7.19)$$

From (7.17), (7.18), (7.19) and the Lebesgue dominated convergence theorem it follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^+} \|G_n(\mu)y(\mu) - G(\mu)y(\mu)\|_{\mathbb{C}^2}^2 \frac{d\mu}{2\pi} = 0.$$

The last equality is the equality (1.9). □

Definition 7.4. Let $G(\mu)$, $\mu \in \mathbb{R}^+$, be a 2×2 matrix function defined almost everywhere on \mathbb{R}^+ . Then the matrix-function $G^{-1}(\mu)$ is defined for those $\mu \in \mathbb{R}^+$ for which the matrix $G(\mu)$ is defined and invertible:

$$G^{-1}(\mu) \stackrel{\text{def}}{=} (G(\mu))^{-1},$$

where $(G(\mu))^{-1}$ is the matrix inverse to the matrix $G(\mu)$.

In particular, if the matrix $G(\mu)$ is invertible for almost every $\mu \in \mathbb{R}^+$, then the matrix function $G^{-1}(\mu)$ is defined almost everywhere.

Theorem 7.1. Let $G(\mu)$, $\mu \in \mathbb{R}^+$, be a matrix function from the set $L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$, (Definition 7.1), and $\mathcal{M}_G : \mathcal{K} \rightarrow \mathcal{K}$ be the operator generated by the matrix function G .

1. In order to the operator \mathcal{M}_G be invertible it is necessary and sufficient that both of the following two conditions are satisfied:
 - a). For almost every $\mu \in \mathbb{R}^+$, the matrix $G(\mu)$ is invertible, i.e. the condition

$$\det G(\mu) \neq 0 \quad \text{for almost every } \mu \in \mathbb{R}^+ \quad (7.20)$$

holds.

- b). The matrix function $G^{-1}(\mu)$, which under the condition (7.20) is defined for almost every $\mu \in \mathbb{R}^+$, belongs to the set $L_{\mathfrak{M}_2}^\infty(\mathbb{R}^+)$:

$$\text{ess sup}_{\mu \in \mathbb{R}^+} \|G^{-1}(\mu)\| < \infty. \quad (7.21)$$

2. If the conditions (7.20) and (7.21) are satisfied, then the inverse operator $(\mathcal{M}_G)^{-1}$ is expressible as

$$(\mathcal{M}_G)^{-1} = \mathcal{M}_{G^{-1}}. \quad (7.22)$$

3. If the condition (7.20) is violated, then the point $\zeta = 0$ belongs to both the point spectrum $\sigma_p((\mathcal{M}_F))$ and the residual spectrum $\sigma_r((\mathcal{M}_G))$ of the operator \mathcal{M}_G .
4. If the condition (7.20) is satisfied, but the condition (7.21) is violated, then the point $\zeta = 0$ belongs to the continuous spectrum $\sigma_c((\mathcal{M}_G))$ of the operator \mathcal{M}_F , but $0 \notin \sigma_p((\mathcal{M}_G))$, $0 \notin \sigma_r((\mathcal{M}_G))$.

Proof.

- Assume that the conditions (7.20) and (7.21) are satisfied, so the

operator $\mathcal{M}_{G^{-1}} : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ exists and is a bounded operator. According to the statements 3 and 1 of Lemma 7.1,

$$\mathcal{M}_{G^{-1}} \cdot \mathcal{M}_G = \mathcal{M}_G \cdot \mathcal{M}_{G^{-1}} = \mathcal{M}_{G^{-1} \cdot G} = \mathcal{M}_{G \cdot G^{-1}} = \mathcal{M}_I = J_{\mathcal{K}}.$$

So the operator \mathcal{M}_G is invertible, and the inverse operator $(\mathcal{M}_G)^{-1}$ is expressed by the equality (7.22).

◦ Assume that the condition (7.20) is violated. Then there exists a measurable set S , $S \in \mathbb{R}^+$, $m(S) > 0$, such that for every $\mu \in S$ there exists a vector $y(\mu) \in \mathbb{C}^2$, $y(\mu) \neq 0$, such that $G(\mu)y(\mu) = 0$. One can choose the vectors $y(\mu)$ in such a way that the function $y(\mu)$, $\mu \in S$, is measurable. One can normalize the vectors $y(\mu)$ in such a way that $\int_S y^*(\mu) y(\mu) \frac{d\mu}{2\pi} = 1$. Until now the function $y(\mu)$ is defined only on S .

Let us extend the function $y(\mu)$ on the whole \mathbb{R}^+ putting $y(\mu) = 0$ for $\mu \in \mathbb{R}^+ \setminus S$. Now the function $y(\mu)$ is defined everywhere on \mathbb{R}^+ and satisfies the condition $\int_{\mathbb{R}^+} y^*(\mu) y(\mu) \frac{d\mu}{2\pi} = 1$, that is

$$y(\mu) \in \mathcal{K}, \quad y \neq 0 \text{ in } \mathcal{K}.$$

From the other hand,

$$G(\mu)y(\mu) = 0 \quad \text{for every } \mu \in \mathbb{R}^+.$$

This means that the vector y , $y \neq 0$, belongs to the null space of the operator \mathcal{M}_G . In other words, the point $\zeta = 0$ belongs to the point spectrum of the operator \mathcal{M}_G . Considering the matrix function $G^*(\mu)$ we obtain that the point $\zeta = 0$ belongs to the point spectrum of the operator $(\mathcal{M}_G)^*$. In other words, the point $\zeta = 0$ belongs to the residual spectrum of the operator \mathcal{M}_G . In particular the operator \mathcal{M}_G is not invertible.

◦ Assume that the condition (7.20) is satisfied. Then the point $\zeta = 0$ belongs neither to the point spectrum, nor to the residual spectrum of the operator \mathcal{M}_G . Indeed let $y \in \mathcal{K}$, $y \neq 0$, but $\mathcal{M}_G y = 0$. Let $S = \{\mu \in \mathbb{R}^+ : y(\mu) \neq 0\}$. Since $y \neq 0$ in \mathcal{K} , $m(S) > 0$. From the other hand, since $\mathcal{M}_G y = 0$ in \mathcal{K} , $G(\mu)y(\mu) = 0$ for almost every $\mu \in \mathbb{R}^+$. Therefore $\det G(\mu) = 0$ for almost every $\mu \in S$. This contradicts the condition (7.20). Thus, $0 \notin \sigma_p(\mathcal{M}_G)$. Analogously $0 \notin \sigma_p(\mathcal{M}_{G^*}) = \overline{\sigma_r(\mathcal{M}_G)}$.

◦ Let the condition (7.20) be satisfied. Then the matrix function $(G(\mu))^{-1}$ is defined for almost all $\mu \in \mathbb{R}^+$. Given $\varepsilon > 0$, let $S_\varepsilon = \{\mu \in$

$\mathbb{R}^+ : \|(G(\mu))^{-1}\| \geq \varepsilon^{-1}\}$. Assume moreover that the condition (7.21) is violated. Then for every $\varepsilon > 0$, the condition $m(S_\varepsilon) > 0$ holds. If $\|(G(\mu))^{-1}\| \geq \varepsilon^{-1}$ for some μ , then there exists a vector $v(\mu) \in \mathbb{C}^2$, $v(\mu) \neq 0$ such that $\|(G(\mu))^{-1}v(\mu)\| \geq \varepsilon^{-1}\|v(\mu)\|$. The vector $y(\mu) = (G(\mu))^{-1}v(\mu)$ satisfies the conditions $y(\mu) \neq 0$, $\|G(\mu)y(\mu)\| \leq \varepsilon\|y(\mu)\|$. Thus if the condition (7.20) is satisfied, but the condition (7.21) is violated, then for every fixed $\varepsilon > 0$ there exists the vector function $y(\mu)$ defined on the set S_ε which satisfies the conditions

$$y(\mu) \neq 0, \quad \|G(\mu)y(\mu)\| \leq \varepsilon\|y(\mu)\| \text{ for every } \mu \in S_\varepsilon, \text{ where } m(S_\varepsilon) > 0.$$

The function $y(\mu)$, defined up to now only on S_ε , can be chosen to be measurable. Normalizing the vectors $y(\mu)$ appropriately, we can satisfy the condition $\int_{S_\varepsilon} y^*(\mu)y(\mu) \frac{d\mu}{2\pi} = 1$. Let us extend the function $y(\mu)$ from the set S_ε to the whole \mathbb{R}^+ putting $y(\mu) = 0$ for $\mu \in \mathbb{R}^+ \setminus S_\varepsilon$. The extended function $y(\mu)$ satisfies the condition $\int_{\mathbb{R}^+} y^*(\mu)y(\mu) \frac{d\mu}{2\pi} = 1$.

In other words,

$$y \in \mathcal{K}, \quad \|y\|_{\mathcal{K}} = 1. \quad (7.23)$$

On the other hand, the inequality $\|G(\mu)y(\mu)\| \leq \varepsilon\|y(\mu)\|$, which holds for the extended function $y(\mu)$ at every $\mu \in \mathbb{R}^+$, implies the inequality

$$\|\mathcal{M}_G y\|_{\mathcal{K}} \leq \varepsilon. \quad (7.24)$$

Thus if the condition (7.20) is satisfied, but the condition (7.21) is violated, then for every $\varepsilon > 0$ there exists $y \in \mathcal{K}$ satisfying the conditions (7.23) and (7.24). Therefore the point $\zeta = 0$ belongs to the continuous spectrum $\sigma_c(\mathcal{M}_G)$ of the operator \mathcal{M}_G . (We already know that if the condition (7.20) is satisfied, then $0 \notin \sigma_p(\mathcal{M}_G)$, $0 \notin \sigma_r(\mathcal{M}_G)$). In particular, the operator \mathcal{M}_G is not invertible. \square

8. The functional model of the operator $\mathcal{F}_{\mathbb{R}^+}$.

In this section we construct the functional model of the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$. First we do formal calculations. Then we justify them.

The operator \mathcal{L} commutes with the operators $\mathcal{F}_{\mathbb{R}^+}$, $\mathcal{F}_{\mathbb{R}^+}^*$. (Theorem 5.1). Let $\mu \in \mathbb{R}^+$. The "eigenspace" of the operator \mathcal{L} corresponding to the eigenvalue $\lambda(\mu)$ is two-dimensional and is generated by the "eigenfunctions" (6.8). Would be the "eigenfunctions" (6.8) of the

operator \mathcal{L} "true" $L^2(\mathbb{R}^+)$ -functions, then the two-dimensional subspace generated by them will be invariant with respect to each of the operators $\mathcal{F}_{\mathbb{R}^+}$ and $\mathcal{F}_{\mathbb{R}^+}^*$. This means that for some matrix

$$F(\mu) = \begin{bmatrix} f_{++}(\mu) & f_{+-}(\mu) \\ f_{-+}(\mu) & f_{--}(\mu) \end{bmatrix}, \quad (8.1)$$

which is constants with respect to t , the equality holds

$$\begin{aligned} (\mathcal{F}_{\mathbb{R}^+} \psi_+(\cdot, \mu))(t) &= \psi_+(t, \mu) f_{++}(\mu) + \psi_-(t, \mu) f_{-+}(\mu), \\ (\mathcal{F}_{\mathbb{R}^+} \psi_-(\cdot, \mu))(t) &= \psi_+(t, \mu) f_{+-}(\mu) + \psi_-(t, \mu) f_{--}(\mu). \end{aligned}$$

The matrix form of these equalities is:

$$(\mathcal{F}_{\mathbb{R}^+} \psi(\cdot, \mu))(t) = \psi(t, \mu) F(\mu). \quad (8.2a)$$

We show that

$$(\mathcal{F}_{\mathbb{R}^+}^* \psi(\cdot, \mu))(t) = \psi(t, \mu) F^*(\mu), \quad (8.2b)$$

where $F^*(\mu)$ is the matrix Hermitian conjugated to the matrix $F(\mu)$.

However the functions $\psi_{\pm}(t, \mu)$ does not belong to $L^2(\mathbb{R}^+)$. So the operators $\mathcal{F}_{\mathbb{R}^+}$, $\mathcal{F}_{\mathbb{R}^+}^*$, considered as an operator acting in $L^2(\mathbb{R}^+)$, are not applicable to the functions $\psi_+(t, \mu)$, $\psi_-(t, \mu)$. Nevertheless we can consider the Fourier integrals $\mathcal{F}_{\mathbb{R}^+} \psi_{\pm}(\cdot, \mu)$, $\mathcal{F}_{\mathbb{R}^+}^* \psi_{\pm}(\cdot, \mu)$ in some *Pickwick sense*. Namely we interpret the expressions $(\mathcal{F}_{\mathbb{R}^+} \psi_{\pm}(\cdot, \mu))(t)$ and $(\mathcal{F}_{\mathbb{R}^+}^* \psi_{\pm}(\cdot, \mu))(t)$ as

$$(\mathcal{F}_{\mathbb{R}^+} \psi_{\pm}(\cdot, \mu))(t) = \lim_{\substack{\varepsilon \rightarrow +0 \\ R \rightarrow +\infty}} \frac{1}{\sqrt{2\pi}} \int_{\varepsilon}^R \xi^{-1/2 \pm i\mu} e^{i\xi t} d\xi, \quad (8.3a)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* \psi_{\pm}(\cdot, \mu))(t) = \lim_{\substack{\varepsilon \rightarrow +0 \\ R \rightarrow +\infty}} \frac{1}{\sqrt{2\pi}} \int_{\varepsilon}^R \xi^{-1/2 \pm i\mu} e^{-i\xi t} d\xi, \quad (8.3b)$$

In (8.3), $t \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$. It turns out that the limits in (8.3) exist and are uniform if t belongs to any fixed interval separated from zero and infinity. (We shall see this when calculating the integrals.) Changing

variable in (8.3a): $\xi \rightarrow \xi/t$, and using the homogeneity properties of the functions $\psi_{\pm}(t, \mu)$ with respect to t , we obtain that

$$(\mathcal{F}_{\mathbb{R}^+} \psi_+(\cdot, \mu))(t) = \psi_-(t, \mu) f_{-+}(\mu), \quad (8.4a)$$

$$(\mathcal{F}_{\mathbb{R}^+} \psi_-(\cdot, \mu))(t) = \psi_+(t, \mu) f_{+-}(\mu), \quad (8.4b)$$

where $t \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$, and

$$f_{-+}(\mu) = \lim_{\substack{\varepsilon \rightarrow +0 \\ R \rightarrow +\infty}} \frac{1}{\sqrt{2\pi}} \int_{\varepsilon}^R \xi^{-1/2+i\mu} e^{i\xi} d\xi, \quad (8.5a)$$

$$f_{+-}(\mu) = \lim_{\substack{\varepsilon \rightarrow +0 \\ R \rightarrow +\infty}} \frac{1}{\sqrt{2\pi}} \int_{\varepsilon}^R \xi^{-1/2-i\mu} e^{i\xi} d\xi. \quad (8.5b)$$

Changing variable in (8.3b): $\xi \rightarrow \xi/t$, and using the homogeneity properties of the functions $\psi_{\pm}(t, \mu)$ with respect to t , we obtain that

$$(\mathcal{F}_{\mathbb{R}^+}^* \psi_+(\cdot, \mu))(t) = \psi_-(t, \mu) \overline{f_{+-}(\mu)}, \quad (8.6a)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* \psi_-(\cdot, \mu))(t) = \psi_+(t, \mu) \overline{f_{-+}(\mu)}, \quad (8.6b)$$

where $t \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$. Let us calculate the integrals in (8.5). These integrals can be presented as

$$\lim_{\substack{\varepsilon \rightarrow +0 \\ R \rightarrow +\infty}} \int_{\varepsilon}^R \xi^{-1/2 \pm i\mu} e^{i\xi} d\xi = e^{i\frac{\pi}{4} \mp \frac{\mu\pi}{2}} \lim_{\substack{\varepsilon \rightarrow +0 \\ R \rightarrow +\infty}} \int_{[-i\varepsilon, iR]} f(\zeta) d\zeta, \quad (8.7)$$

where

$$f(\zeta) = \zeta^{-1/2 \pm i\mu} e^{-\zeta}, \quad \arg \zeta > 0 \quad \text{for } \zeta \in (0, \infty). \quad (8.8)$$

Then we ‘rotate’ the ray of integration from the ray $(0, -i\infty)$ to the ray $(0, \infty)$. The function $f(\zeta)$ is holomorphic in the domain $\mathbb{C} \setminus (-\infty, 0]$. According to Cauchy integral theorem,

$$\int_{[-i\varepsilon, iR]} f(\zeta) d\zeta = \int_{[\varepsilon, R]} f(\zeta) d\zeta + \int_{\gamma_{\varepsilon}} f(\zeta) d\zeta + \int_{\gamma_R} f(\zeta) d\zeta,$$

where γ_{ε} and γ_R are the arcs $-\pi/2 \leq \arg z \leq 0$, $|z| = \varepsilon$ and $|z|$ respectively. The functions $f(\zeta)$ grows as $|\varepsilon|^{-1/2}$ as $\zeta \in \gamma_{\varepsilon}$, $\varepsilon \rightarrow 0$, and the length of the arc γ_{ε} decays as ε , as $\varepsilon \rightarrow 0$. Therefore, $\int_{\gamma_{\varepsilon}} f(\zeta) d\zeta \rightarrow 0$

as $\varepsilon \rightarrow 0$. Applying Jordan lemma to the function $f(\zeta)$ in the quadrant $-\pi/2 \leq \arg \zeta \leq 0$, we conclude that $\int_{\gamma_R} f(\zeta) d\zeta \rightarrow 0$ as $R \rightarrow \infty$.

Therefore

$$\lim_{\substack{\varepsilon \rightarrow +0 \\ R \rightarrow +\infty}} \int_{\varepsilon}^R \xi^{-1/2 \pm i\mu} e^{i\xi} d\xi = e^{i\frac{\pi}{4} \mp \frac{\mu\pi}{2}} \int_0^{+\infty} \xi^{-1/2 \pm i\mu} e^{-\xi} d\xi.$$

The integral in the right hand side of the last formula is the Euler integral representing the Γ -function. Thus

$$\lim_{\substack{\varepsilon \rightarrow +0 \\ R \rightarrow +\infty}} \int_{\varepsilon}^R \xi^{-1/2 \pm i\mu} e^{i\xi} d\xi = e^{i\frac{\pi}{4} \mp \frac{\mu\pi}{2}} \Gamma(1/2 \pm i\mu), \quad -\infty < \mu < \infty, \quad (8.9)$$

and

$$f_{+-}(\mu) = \frac{1}{\sqrt{2\pi}} e^{i\frac{\pi}{4} + \frac{\mu\pi}{2}} \Gamma(1/2 - i\mu), \quad (8.10a)$$

$$f_{-+}(\mu) = \frac{1}{\sqrt{2\pi}} e^{i\frac{\pi}{4} - \frac{\mu\pi}{2}} \Gamma(1/2 + i\mu). \quad (8.10b)$$

So, the matrix $F(\mu) = \begin{bmatrix} f_{++}(\mu) & f_{+-}(\mu) \\ f_{-+}(\mu) & f_{--}(\mu) \end{bmatrix}$ in (8.2) is of the form

$$F(\mu) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2\pi}} e^{i\frac{\pi}{4} + \frac{\mu\pi}{2}} \Gamma(1/2 - i\mu) \\ \frac{1}{\sqrt{2\pi}} e^{i\frac{\pi}{4} - \frac{\mu\pi}{2}} \Gamma(1/2 + i\mu) & 0 \end{bmatrix} \quad (8.11)$$

Thus the equalities (8.2) hold with the matrix $F(\mu)$ of the form (8.11). Given $x(t) \in L^2(\mathbb{R}^+)$, we apply the operators $\mathcal{F}_{\mathbb{R}^+}$, $\mathcal{F}_{\mathbb{R}^+}^*$ to the spectral expansion (6.29a), (6.29b). Applying the operators $\mathcal{F}_{\mathbb{R}^+}$, $\mathcal{F}_{\mathbb{R}^+}^*$ to the linear combination $\psi(t, \mu) \hat{x}(\mu)$, we should take into account that these operators act on functions of variable t and the coefficients $\hat{x}(\mu)$ of this linear combination do not depend on t . Therefore

$$\mathcal{F}_{\mathbb{R}^+}(\psi(\cdot, \mu) \hat{x}(\mu))(t) = (\mathcal{F}_{\mathbb{R}^+} \psi(\cdot, \mu))(t) \hat{x}(\mu), \quad (8.12a)$$

$$\mathcal{F}_{\mathbb{R}^+}^*(\hat{x}(\mu) \psi(\cdot, \mu))(t) = (\mathcal{F}_{\mathbb{R}^+}^* \psi(\cdot, \mu))(t) \hat{x}(\mu). \quad (8.12b)$$

Carry the operator $\mathcal{F}_{\mathbb{R}^+}$ through the integral in (6.29b) and using (8.12), we obtain

$$(\mathcal{F}_{\mathbb{R}^+} x)(t) = \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) u_{\mathcal{F}_{\mathbb{R}^+}}(\mu) \frac{d\mu}{2\pi}, \quad (\mathcal{F}_{\mathbb{R}^+}^* x)(t) = \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) u_{\mathcal{F}_{\mathbb{R}^+}^*}(\mu) \frac{d\mu}{2\pi}, \quad (8.13a)$$

where

$$u_{\mathcal{F}_{\mathbb{R}^+}}(\mu) = F(\mu) \hat{x}(\mu), \quad u_{\mathcal{F}_{\mathbb{R}^+}^*}(\mu) = F^*(\mu) \hat{x}(\mu). \quad (8.13b)$$

Let us go to prove rigorously the formulas (8.13) expressing the spectral resolution of the vectors $\mathcal{F}_{\mathbb{R}^+} x$ in terms of the spectral resolution (6.29a) of the vector x . In this proof we use the following expressions for the absolute values of the entries of the matrix $F(\mu)$:

$$|f_{+-}(\mu)| = (1 + e^{-2\pi\mu})^{-1/2}, \quad |f_{-+}(\mu)| = (1 + e^{2\pi\mu})^{-1/2}, \quad \mu \in \mathbb{R}^+. \quad (8.14)$$

The expressions (8.14) are derived from (8.10). Since

$$\Gamma(1/2 + i\mu)\Gamma(1/2 - i\mu) = \frac{\pi}{\cosh \pi\mu} \quad (8.15)$$

and the numbers $\Gamma(1/2 \pm i\mu)$ are complex conjugated, then

$$|\Gamma(1/2 \pm i\mu)|^2 = \frac{2\pi}{e^{\pi\mu} + e^{-\pi\mu}}, \quad \mu \in \mathbb{R}^+. \quad (8.16)$$

The equalities (8.14) follows from the last formula and from (8.10). We remark that in particular

$$1/\sqrt{2} < |f_{+-}(\mu)| < 1, \quad |f_{-+}(\mu)| < 1/\sqrt{2}, \quad \mu \in \mathbb{R}^+. \quad (8.17)$$

If μ runs over the interval $[0, \infty)$, then $|f_{+-}(\mu)|$ increases from $2^{-1/2}$ to 1 and $|f_{-+}(\mu)|$ decreases from $2^{-1/2}$ to 0. In particular,

$$\sup_{\mu \in \mathbb{R}^+} |f_{+-}(\mu)| = \text{ess sup}_{\mu \in \mathbb{R}^+} |f_{+-}(\mu)| = 1. \quad (8.18)$$

From (8.10) and (8.16) it follows that

$$|f_{+-}(\mu)|^2 + |f_{-+}(\mu)|^2 = 1, \quad (8.19)$$

$$|f_{+-}(\mu)| + |f_{-+}(\mu)| = \sqrt{1 + \frac{1}{\cosh \pi\mu}}, \quad (8.20)$$

thus

$$1 \leq |f_{+-}(\mu)| + |f_{-+}(\mu)| \leq \sqrt{2}, \quad 0 \leq \mu < \infty. \quad (8.21)$$

In view of the diagonal structure (8.11) of the matrix $F(\mu)$ and the estimates (8.17), (8.18) for its entries, the equalities

$$\|F(\mu)\| < 1 \quad \forall \mu \in (0, \infty) \quad (8.22a)$$

and

$$\operatorname{ess\,sup}_{\mu \in \mathbb{R}^+} \|F(\mu)\| = 1. \quad (8.22b)$$

hold.

Theorem 8.1. *Let $x(t) \in L^2(\mathbb{R}^+)$, and $\hat{x}(\mu)$ be the Fourier transform of x , (6.29a):*

$$\hat{x}(\mu) = \int_{\xi \in \mathbb{R}^+} \psi^*(\xi, \mu) x(\xi) d\xi, \quad \mu \in \mathbb{R}^+.$$

Then the Fourier transforms $u_{\mathcal{F}_E}(\mu)$, $u_{\mathcal{F}_E^}(\mu)$ of the functions $(\mathcal{F}_{\mathbb{R}^+} x)(t)$, $(\mathcal{F}_{\mathbb{R}^+}^* x)(t)$:*

$$u_{\mathcal{F}_E}(\mu) = \int_{\xi \in \mathbb{R}^+} \psi^*(\xi, \mu) (\mathcal{F}_{\mathbb{R}^+} x)(\xi) d\xi, \quad u_{\mathcal{F}_E^*}(\mu) = \int_{\xi \in \mathbb{R}^+} \psi^*(\xi, \mu) (\mathcal{F}_{\mathbb{R}^+}^* x)(\xi) d\xi \quad (8.23)$$

are expressed in terms of $\hat{x}(\mu)$ by the formula (8.13b).

The functions $(\mathcal{F}_{\mathbb{R}^+} x)(t)$, $(\mathcal{F}_{\mathbb{R}^+}^ x)(t)$ are expressed by the formula (8.13a):*

$$\begin{aligned} (\mathcal{F}_{\mathbb{R}^+} x)(t) &= \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) F(\mu) \hat{x}(\mu) \frac{d\mu}{2\pi}, \\ (\mathcal{F}_{\mathbb{R}^+}^* x)(t) &= \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) F^*(\mu) \hat{x}(\mu) \frac{d\mu}{2\pi}. \end{aligned}$$

Proof. We substitute the expression

$$x(\xi) = \int_{\mu \in \mathbb{R}^+} \hat{x}(\mu) \psi(\xi, \mu) \frac{d\mu}{2\pi}$$

for the function x , (6.29b), into the formulas (1.1) and (1.2) which defines the truncated Fourier operator $\mathcal{F}_{\mathbb{R}^+}$ and the adjoint operator $\mathcal{F}_{\mathbb{R}^+}^*$. To carry the operators $\mathcal{F}_{\mathbb{R}^+}$, $\mathcal{F}_{\mathbb{R}^+}^*$ through the integral in (6.29b), we have to change the order of integration in the iterated integrals

$$(\mathcal{F}_{\mathbb{R}^+} x)(t) = \int_{\xi \in \mathbb{R}^+} \left(\int_{\mu \in \mathbb{R}^+} \hat{x}(\mu) \psi(\xi, \mu) \frac{d\mu}{2\pi} \right) e^{it\xi} d\xi, \quad (8.24a)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* x)(t) = \int_{\xi \in \mathbb{R}^+} \left(\int_{\mu \in \mathbb{R}^+} \hat{x}(\mu) \psi(\xi, \mu) \frac{d\mu}{2\pi} \right) e^{-it\xi} d\xi. \quad (8.24b)$$

Usual tool to justify the change of the order of integration is the Fubini theorem. However the Fubini theorem is not applicable to the iterated integrals (8.24). The function under the integral is not summable with respect to ξ .

To curry the operators $\mathcal{F}_{\mathbb{R}^+}$, $\mathcal{F}_{\mathbb{R}^+}^*$ through the integral in (6.29b), we use a regularization procedure. Given $\varepsilon > 0$, we define *the regularization operator* $\mathcal{R}_\varepsilon : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$,

$$\mathcal{R}_\varepsilon x(t) = e^{-\varepsilon t} x(t), \quad \forall x \in L^2(\mathbb{R}^+). \quad (8.25)$$

It is clear that for every $x \in L^2(\mathbb{R}^+)$,

$$\|\mathcal{R}_\varepsilon x - x\|_{L^2(\mathbb{R}^+)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow +0.$$

The kernel of the operator $\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon$ can be calculated without difficulties. Let $x \in L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$. Then

$$\begin{aligned} (\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon x)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\xi \in \mathbb{R}^+} e^{i\xi s} e^{-\varepsilon \xi} x(\xi) d\xi, \\ (\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon x)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\xi \in \mathbb{R}^+} e^{-i\xi s} e^{-\varepsilon \xi} x(\xi) d\xi, \end{aligned}$$

Substituting the expression (6.29b) for the function $x(\xi)$ into the last formula, we present the functions $(\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon x)(t)$, $(\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon x)(t)$ as the iterated integrals

$$(\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\xi \in \mathbb{R}^+} e^{i\xi(t+i\varepsilon)} \left(\int_{\mu \in \mathbb{R}^+} \psi(\xi, \mu) \hat{x}(\mu) \frac{d\mu}{2\pi} \right) d\xi, \quad (8.26a)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon x)(t) = \frac{1}{\sqrt{2\pi}} \int_{\xi \in \mathbb{R}^+} e^{-i\xi(t-i\varepsilon)} \left(\int_{\mu \in \mathbb{R}^+} \psi(\xi, \mu) \hat{x}(\mu) \frac{d\mu}{2\pi} \right) d\xi, \quad (8.26b)$$

We assume firstly that the function $\hat{x}(\mu)$ belongs to $L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$. The Fubini theorem is applicable to each of the iterated integral (8.26).

So for every fixed $\varepsilon > 0$ we can change the order of integration there. Changing the order, we obtain

$$(\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon x)(t) = \int_{\mu \in \mathbb{R}^+} (\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon \psi(\cdot, \mu))(t) \hat{x}(\mu) \frac{d\mu}{2\pi}, \quad (8.27a)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon x)(t) = \int_{\mu \in \mathbb{R}^+} (\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon \psi(\cdot, \mu))(t) \hat{x}(\mu) \frac{d\mu}{2\pi}, \quad (8.27b)$$

where

$$(\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon \psi_\pm(\cdot, \mu))(t) = \frac{1}{\sqrt{2\pi}} \int_{\xi \in \mathbb{R}^+} \xi^{-1/2 \pm i\mu} e^{i(t+i\varepsilon)\xi} d\xi, \quad (8.28a)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon \psi_\pm(\cdot, \mu))(t) = \frac{1}{\sqrt{2\pi}} \int_{\xi \in \mathbb{R}^+} \xi^{-1/2 \pm i\mu} e^{-i(t-i\varepsilon)\xi} d\xi. \quad (8.28b)$$

The integrals in (8.28) can be calculated explicitly:

$$(\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon \psi_+(\cdot, \mu))(t) = \psi_-(t + i\varepsilon, \mu) f_{-+}(\mu), \quad (8.29a)$$

$$(\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon \psi_-(\cdot, \mu))(t) = \psi_+(t + i\varepsilon, \mu) f_{+-}(\mu), \quad (8.29b)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon \psi_+(\cdot, \mu))(t) = \psi_-(t - i\varepsilon, \mu) \overline{f_{+-}(\mu)}, \quad (8.29c)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon \psi_-(\cdot, \mu))(t) = \psi_+(t - i\varepsilon, \mu) \overline{f_{-+}(\mu)}, \quad (8.29d)$$

where $f_{+-}(\mu)$ and $f_{-+}(\mu)$ are the same that in (8.10), and

$$\psi_+(t \pm i\varepsilon, \mu) = (t \pm i\varepsilon)^{-1/2 + i\mu} = e^{(-1/2 + i\mu)(\ln |t+i\varepsilon| \pm i \arg(t+i\varepsilon))}, \quad (8.30a)$$

$$\psi_-(t \pm i\varepsilon, \mu) = (t \pm i\varepsilon)^{-1/2 - i\mu} = e^{(-1/2 - i\mu)(\ln |t+i\varepsilon| \pm i \arg(t+i\varepsilon))}. \quad (8.30b)$$

Here

$$0 < \arg(t + i\varepsilon) < \pi/2 \quad \text{for } t > 0, \varepsilon > 0. \quad (8.31)$$

Formulas (8.30) are derived similarly to formulas (8.4). We change variable: $\xi \rightarrow \xi/|t + i\varepsilon|$, and then rotate the ray of integration. From (8.30) and (8.31) it follows that for every $t \in (0, \infty)$, $\mu \in (0, \infty)$

$$|\psi_+(t + i\varepsilon, \mu)| \leq t^{-1/2}, \quad |\psi_-(t + i\varepsilon, \mu)| \leq t^{-1/2} e^{\mu\pi/2}, \quad (8.32a)$$

$$|\psi_+(t - i\varepsilon, \mu)| \leq t^{-1/2} e^{\mu\pi/2}, \quad |\psi_-(t - i\varepsilon, \mu)| \leq t^{-1/2}. \quad (8.32b)$$

Taking into account the estimates (8.14), we obtain from (8.29) and (8.32) the estimates

$$|(\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon \psi_\pm(\cdot, \mu))(t)| \leq t^{-1/2}, \quad (8.33a)$$

$$|(\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon \psi_\pm(\cdot, \mu))(t)| \leq t^{-1/2}, \quad (8.33b)$$

which hold for every $\mu > 0$, $t > 0$ and $\varepsilon > 0$. In particular, the expressions in the right hand sides of (8.33) do not depend on ε . Moreover, from (8.29) it follows that for every fixed $\mu > 0$ and $t > 0$, there exist the limits

$$\lim_{\varepsilon \rightarrow +0} (\mathcal{F}_E \mathcal{R}_\varepsilon \psi(\cdot, \mu))(t) = \psi(t, \mu) F(\mu), \quad (8.34a)$$

$$\lim_{\varepsilon \rightarrow +0} (\mathcal{F}_E^* \mathcal{R}_\varepsilon \psi(\cdot, \mu))(t) = \psi(t, \mu) F^*(\mu). \quad (8.34b)$$

Using the Lebesgue dominating convergence theorem, we conclude that for every fixed $t > 0$

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \int_{\mu \in \mathbb{R}^+} (\mathcal{F}_E \mathcal{R}_\varepsilon \psi(\cdot, \mu))(t) \hat{x}(\mu) d\mu &= \int_{\mu \in \mathbb{R}^+} (\psi(t, \mu) F(\mu)) \hat{x}(\mu) d\mu, \\ \lim_{\varepsilon \rightarrow +0} \int_{\mu \in \mathbb{R}^+} (\mathcal{F}_E^* \mathcal{R}_\varepsilon \psi(\cdot, \mu))(t) \hat{x}(\mu) d\mu &= \int_{\mu \in \mathbb{R}^+} (\psi(t, \mu) F^*(\mu)) \hat{x}(\mu) d\mu. \end{aligned}$$

Involving (8.27a), we see that

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} (\mathcal{F}_E \mathcal{R}_\varepsilon x)(t) &= \int_{\mu \in \mathbb{R}^+} (\psi(t, \mu) F(\mu)) \hat{x}(\mu) \frac{d\mu}{2\pi}, \\ \lim_{\varepsilon \rightarrow +0} (\mathcal{F}_E^* \mathcal{R}_\varepsilon x)(t) &= \int_{\mu \in \mathbb{R}^+} (\psi(t, \mu) F^*(\mu)) \hat{x}(\mu) \frac{d\mu}{2\pi}. \end{aligned}$$

for every fixed $t > 0$. From the other hand,

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \|(\mathcal{F}_{\mathbb{R}^+} \mathcal{R}_\varepsilon x)(t) - (\mathcal{F}_{\mathbb{R}^+} x)(t)\|_{L^2(\mathbb{R}^+)} &= 0, \\ \lim_{\varepsilon \rightarrow +0} \|(\mathcal{F}_{\mathbb{R}^+}^* \mathcal{R}_\varepsilon x)(t) - (\mathcal{F}_{\mathbb{R}^+}^* x)(t)\|_{L^2(\mathbb{R}^+)} &= 0, \end{aligned}$$

Comparing the last formulas, we obtain that

$$(\mathcal{F}_{\mathbb{R}^+} x)(t) = \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) F(\mu) \hat{x}(\mu) \frac{d\mu}{2\pi}, \quad (8.35a)$$

$$(\mathcal{F}_{\mathbb{R}^+}^* x)(t) = \int_{\mu \in \mathbb{R}^+} \psi(t, \mu) F^*(\mu) \hat{x}(\mu) \frac{d\mu}{2\pi}, \quad (8.35b)$$

for every $x(t) \in L^2(\mathbb{R}^+)$ for which $\hat{x}(\mu) \in L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$. Since the set $L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is dense in $L^2(\mathbb{R}^+)$, the last equality can be extended to all $x(t) \in L^2(\mathbb{R}^+)$. To justify such extension, one should involve the Parseval equality taking into account that the matrix $F(\mu)$ is bounded, (8.14). \square

9. Spectrum and resolvent of the operator \mathcal{M}_F .

Theorem 8.1 claims that the operator $\mathcal{F}_{\mathbb{R}^+} : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ is unitary equivalent to the matrix multiplication operator \mathcal{M}_F in the model space \mathcal{K} , where $F(\mu)$ is the matrix function of the form (8.11):

$$\mathcal{F}_{\mathbb{R}^+} = U^{-1} \mathcal{M}_F U. \quad (9.1)$$

The unitary operator U and its inverse U^{-1} are expressed⁴ by the equalities (6.26) and (6.27).

The unitary equivalence (9.1) allows to reduce a study of the spectrum and the resolvent of the operator $\mathcal{F}_{\mathbb{R}^+} : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$ to the spectral analysis of the operator $\mathcal{M}_F : \mathcal{K} \rightarrow \mathcal{K}$.

To perform a spectral analysis of the operator \mathcal{M}_F , acting in the *infinite dimensional* space \mathcal{K} , we have to perform the spectral analysis of the 2×2 matrix $F(\mu)$, acting in the *2 dimensional* space \mathbb{C}^2 . The spectral analysis of the matrix $F(\mu)$ can be done for *each* $\mu \in \mathbb{R}^+$ separately. Then we can *glue* the spectrum $\sigma(\mathcal{M}_F)$ of the operator \mathcal{M}_F from the spectra $\sigma(F(\mu))$ of the matrices $F(\mu)$, as well as the resolvent of the operator \mathcal{M}_F from the resolvents of the matrices $F(\mu)$:

$$\sigma(\mathcal{M}_F) = \left(\bigcup_{\mu \in [0, \infty)} \sigma(F(\mu)) \right) \bigcup \{0\}. \quad (9.2)$$

$$(z\mathcal{I} - \mathcal{M}_F)^{-1} = (\mathcal{M}_{(zI-F)})^{-1} = \mathcal{M}_{(zI-F)^{-1}}. \quad (9.3)$$

(The point $\{0\}$, which appears in the right hand side of (9.2), corresponds to the value $\mu = \infty$.)

⁴In the spectral language, the operators U and U^{-1} are direct and inverse expansions in eigenfunctions of the self-adjoint differential operator \mathcal{L} , which commutes with $\mathcal{F}_{\mathbb{R}^+}$. In the integral transforms language, U and U^{-1} are direct and inverse Mellin transforms restricted on the vertical line $\text{Im } \zeta = \frac{1}{2}$. (See Remark 6.1.)

Therefore we have first to perform a spectral analysis of the matrix $F(\mu)$ for each μ separately.
The spectrum of the matrix $F(\mu)$. According to (8.11), the matrix $zI - F(\mu)$ is of the form

$$zI - F(\mu) = \begin{bmatrix} z & -f_{+-}(\mu) \\ -f_{-+}(\mu) & z \end{bmatrix} \quad (9.4)$$

where f_{-+} , f_{+-} are of the form (8.10). According to the identity (8.15),

$$f_{+-}(\mu) \cdot f_{-+}(\mu) = \frac{i}{2 \cosh \pi \mu}. \quad (9.5)$$

Let $D(z, \mu)$ be the determinant of the matrix $zI - F(\mu)$:

$$D(z, \mu) = \det(zI - F(\mu)). \quad (9.6)$$

From (9.4) and (9.5) it follows that

$$D(z, \mu) = z^2 - \frac{i}{2 \cosh \pi \mu}. \quad (9.7)$$

For $\mu \in [0, \infty)$, let

$$\zeta(\mu) = e^{i\pi/4} \frac{1}{\sqrt{2} \cosh \pi \mu}, \quad (9.8)$$

so (9.5) takes form

$$\zeta^2(\mu) = f_{-+}(\mu) f_{+-}(\mu). \quad (9.9)$$

Let us denote the roots of the characteristic polynomial $D(z, \mu)$ of the matrix $F(\mu)$:

$$D(z, \mu) = (z - \zeta_+(\mu)) \cdot (z - \zeta_-(\mu)), \quad (9.10)$$

where

$$\zeta_+(\mu) = \zeta(\mu), \quad \zeta_-(\mu) = -\zeta(\mu). \quad (9.11)$$

It is clear that $\zeta(\mu) \neq 0$, so $\zeta_+(\mu) \neq \zeta_-(\mu)$ for every $\mu \in [0, \infty)$.

Lemma 9.1.

1. For $\mu \in [0, \infty)$, the spectrum $\sigma(F(\mu))$ of the matrix $F(\mu)$, (8.11), is simple, and consists of two different points $\zeta_+(\mu)$ and $\zeta_-(\mu)$: (9.11), (9.8):

$$\sigma(F(\mu)) = \{\zeta_+(\mu), \zeta_-(\mu)\}. \quad (9.12)$$

2. If $\mu_1, \mu_2 \in [0, \infty)$, $\mu_1 \neq \mu_2$, then $\sigma(F(\mu_1)) \cap \sigma(F(\mu_2)) = \emptyset$.

The resolvent of the matrix $F(\mu)$.

From (9.4) and the rule of inversion of a 2×2 matrix it follows:

Lemma 9.2.

1. Given $\mu \in [0, \infty)$, the matrix $(zI - F(\mu))^{-1}$ is:

$$(zI - F(\mu))^{-1} = \frac{1}{D(z, \mu)} \begin{bmatrix} z & f_{-+}(\mu) \\ f_{+-}(\mu) & z \end{bmatrix} \quad (9.13)$$

2. The matrix function $(zI - F(\mu))^{-1}$, the resolvent of the matrix $F(\mu)$, is a rational matrix function of z . The only singularities of this matrix function in the extended complex plane $\mathbb{C}_{\text{ext}} = \mathbb{C} \cup \infty$ are simple poles at the points $z = \zeta_+(\mu)$ and $z = \zeta_-(\mu)$:

$$(zI - F(\mu))^{-1} = \frac{E_+(\mu)}{z - \zeta_+(\mu)} + \frac{E_-(\mu)}{z - \zeta_-(\mu)}, \quad (9.14)$$

where the residue matrices $E_+(\mu), E_-(\mu)$ are projector matrices of rank one.

The fact that the residues matrices $E_+(\mu), E_-(\mu)$ are projector matrices of rank one can be obtained analyzing the explicit expression of these matrices. The identity (9.9) has been used by such analysis. We do this analysis later. (See the formulae (10.6)-(10.10)). This can also be derived from general facts about resolvents of operators.

Lemma 9.3. The norm of an arbitrary 2×2 matrix M ,

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},$$

considered as an operator from \mathbb{C}^2 to \mathbb{C}^2 , admits the estimates from above and from below

$$\frac{1}{2} \text{trace}(M^* M) \leq \|M\|^2 \leq \text{trace}(M^* M). \quad (9.15)$$

Assuming that $\det M \neq 0$, the norm of the inverse matrix M^{-1} can be estimated as follows:

$$\begin{aligned} |(\det M)|^{-2} \text{trace}(M^* M) - \frac{2}{\text{trace}(M^* M)} &\leq \\ &\leq \|M^{-1}\|^2 \leq |(\det M)|^{-2} \text{trace}(M^* M) \end{aligned} \quad (9.16)$$

where

$$\text{trace } M^*M = |m_{11}|^2 + |m_{12}|^2 + |m_{21}|^2 + |m_{22}|^2. \quad (9.17)$$

Proof. Let s_0 and s_1 be singular values of the matrix M , that is

$$0 < s_1 \leq s_0, \quad (9.18)$$

and the numbers s_0^2, s_1^2 are eigenvalues of the matrix M^*M . Then

$$\begin{aligned} \|M\| &= s_0, \quad \|M^{-1}\| = s_1^{-1}, \\ \text{trace}(M^*M) &= s_0^2 + s_1^2, \quad |\det(M)|^2 = \det(M^*M) = s_0^2 \cdot s_1^2. \end{aligned}$$

Therefore the inequality (9.15) takes the form

$$\frac{1}{2}(s_0^2 + s_1^2) \leq s_0^2 \leq (s_0^2 + s_1^2),$$

and the inequality (9.16) takes the form

$$(s_0 s_1)^{-2}(s_0^2 + s_1^2) - \frac{2}{s_0^2 + s_1^2} \leq s_1^{-2} \leq (s_0 s_1)^{-2}(s_0^2 + s_1^2).$$

The last inequalities hold for arbitrary numbers s_0, s_1 which satisfy the inequalities (9.18). \square

Lemma 9.4. *For every $\mu \in [0, \infty)$ and every $z \in \mathbb{C} \setminus \sigma(F(\mu))$ the matrix $(zI - F(\mu))^{-1}$ admits the estimates*

$$\begin{aligned} |D(z, \mu)|^{-2}(2|z|^2 + 1) - \frac{2}{2|z|^2 + 1} &\leq \\ &\leq \|(zI - F(\mu))^{-1}\|^2 \leq |D(z, \mu)|^{-2}(2|z|^2 + 1), \end{aligned} \quad (9.19)$$

where $D(z, \mu) = \det(zI - F(\mu))$: (9.6), (9.10), and $\sigma(F(\mu))$ is the spectrum of the matrix $F(\mu)$: (9.12), (9.11), (9.8).

Proof. We apply the estimate (9.16) to the matrix $M = zI - F(\mu)$. According to (9.4), (9.17) and (8.19),

$$\text{trace}((zI - F(\mu))^{-1})^*(zI - F(\mu))^{-1} = 2|z|^2 + 1.$$

\square

The spectrum of the operator \mathcal{M}_F .

Let us describe the spectrum $\sigma(\mathcal{M}_F)$ of the multiplication operator \mathcal{M}_F , where $F(\mu)$ is the matrix function appeared in (8.11).

When μ runs over the interval $[0, \infty)$, the points $\zeta_+(\mu)$ fill the interval $\left(0, \frac{1}{\sqrt{2}} e^{i\pi/4}\right]$ and the points $\zeta_-(\mu)$ fill the interval $\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right)$. When μ increases, the points $\zeta_+(\mu)$, $\zeta_-(\mu)$ move monotonically, so the mappings $\mu \rightarrow \zeta_+(\mu)$ is a homeomorphism of $[0, \infty)$ onto $\left(0, \frac{1}{\sqrt{2}} e^{i\pi/4}\right]$ and the mapping $\mu \rightarrow \zeta_-(\mu)$ is a homeomorphism of $[0, \infty)$ onto $\left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right)$.

Lemma 9.5. *The multiplication operator \mathcal{M}_F , where $F(\mu)$ is the matrix function from (8.11), has neither the point nor the residual spectrum.*

Proof. Let z be a complex number. Then $\mathcal{M}_F - z\mathcal{I} = \mathcal{M}_{F-zI}$. According to (9.6), (9.10), the determinant $\det(F(\mu) - zI)$ does not vanish for $\mu \in [0, \infty)$ if $z \notin \left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right) \cup \left(0, \frac{1}{\sqrt{2}} e^{i\pi/4}\right]$ and vanishes precisely in one point $\mu(z)$ if $z \in \left[-e^{i\pi/4} \frac{1}{\sqrt{2}}, 0\right) \cup \left(0, \frac{1}{\sqrt{2}} e^{i\pi/4}\right]$. In each of these two cases, $m(\{\mu : \det(F(\mu) - zI) = 0\}) = 0$. According to Theorem 7.1, $0 \notin \sigma_p(\mathcal{M}_F - z\mathcal{I})$, $0 \notin \sigma_r(\mathcal{M}_F - z\mathcal{I})$, that is $z \notin \sigma_p(\mathcal{M}_F)$, $z \notin \sigma_r(\mathcal{M}_F)$. \square

Lemma 9.6. *The continuous spectrum $\sigma_c(\mathcal{M}_F)$ of the multiplication operator \mathcal{M}_F , where $F(\mu)$ is the matrix function from (8.11), is the interval $\left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4}\right]$.*

Proof. When μ runs over the interval $[0, \infty)$, the complex number $\frac{i}{2 \cosh \pi \mu}$, which appears in the right hand side of the equality (9.7), fill the interval $(0, i/2]$. Therefore

$$\inf_{\mu \in (0, \infty)} |D(z, \mu)| = \text{dist}(z^2, [0, i/2]) \quad (9.20)$$

In particular,

$$\left(\inf_{\mu \in [0, \infty)} |D(z, \mu)| > 0 \right) \Leftrightarrow (z^2 \notin [0, i/2]),$$

or, what is the same,

$$\left(\inf_{\mu \in [0, \infty)} |D(z, \mu)| > 0 \right) \Leftrightarrow \left(z \notin \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right] \right), \quad (9.21)$$

From the inequalities (9.19) it follows that

$$\begin{aligned} \frac{2|z|^2 + 1}{\left(\inf_{\mu \in [0, \infty)} |D(z, \mu)|\right)^2} - \frac{2}{2|z|^2 + 1} &\leq \\ &\leq \sup_{\mu \in [0, \infty)} \|(zI - F(\mu))^{-1}\|^2 \leq \frac{2|z|^2 + 1}{\left(\inf_{\mu \in [0, \infty)} |D(z, \mu)|\right)^2} \end{aligned} \quad (9.22)$$

From (9.21) and (9.22) it follows that

$$\left(\sup_{\mu \in [0, \infty)} \|(zI - F(\mu))^{-1}\| < \infty \right) \Leftrightarrow \left(z \notin \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right] \right) \quad (9.23)$$

Lemma 9.6 is a consequence of (9.23) and Theorem 7.1. (Theorem 7.1 should be applied to the matrix function $G(z) = zI - F(\mu)$.) \square

From Lemmas (9.5), 9.6 and Statement 2 of Theorem 7.1 we derive:

Theorem 9.1.

1. *The spectrum $\sigma(\mathcal{M}_F)$ of the multiplication operator \mathcal{M}_F , where $F(\mu)$ is the matrix function appeared in (8.11), is:*

$$\sigma(\mathcal{M}_F) = \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right] \quad (9.24)$$

2. *If z does not belong to the spectrum $\sigma(\mathcal{M}_F)$, then the resolvent $(z\mathcal{J} - \mathcal{M}_F)^{-1}$ can be expressed as*

$$(z\mathcal{J} - \mathcal{M}_F)^{-1} = \mathcal{M}_{(zI - F)^{-1}}. \quad (9.25)$$

Estimate for the resolvent of the operator \mathcal{M}_F .

Let z does not belong to the spectrum $\sigma(\mathcal{M}_F)$ of the operator \mathcal{M}_F . According to Statement 2 of Theorem 7.1, the resolvent $(z\mathcal{J} - \mathcal{M}_F)^{-1}$ of this operator can be expressed by (9.25). According to Statement 5 of Theorem 7.1, applied to the matrix function $G(\mu) = (zI - F(\mu))^{-1}$, the estimate

$$\|\mathcal{M}_{(zI - F)^{-1}}\|_{\mathcal{X} \rightarrow \mathcal{X}} = \sup_{\mu \in [0, \infty)} \|(zI - F(\mu))^{-1}\| \quad (9.26)$$

holds. Comparing (9.25), (9.26), (9.22) and (9.20), we obtain two sided estimates for the norm of the resolvent $(z\mathcal{J} - \mathcal{F}_E)^{-1}$ in term of the value

$\text{dist}(z^2, [0, i/2])$:

$$\begin{aligned} \|(z\mathcal{J} - \mathcal{F}_E)^{-1}\| &\leq \frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])}, \quad (9.27a) \\ \frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])} \sqrt{1 - \frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2}} &\leq \|(z\mathcal{J} - \mathcal{F}_E)^{-1}\|. \end{aligned}$$

The value under the square root in (9.27b) is positive since

$$\frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2} \leq \frac{2|z|^2}{(2|z|^2 + 1)^2} \leq \frac{1}{2}.$$

Since $(1 - \alpha) \leq \sqrt{1 - \alpha}$ for $0 \leq \alpha \leq 1$, then

$$1 - \frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2} \leq \sqrt{1 - \frac{2\text{dist}^2(z^2, [0, i/2])}{(2|z|^2 + 1)^2}}.$$

Thus, the lower estimate for the norm of resolvent is

$$\frac{(2|z|^2 + 1)^{1/2}}{\text{dist}(z^2, [0, i/2])} - \frac{2\text{dist}(z^2, [0, i/2])}{(2|z|^2 + 1)^{3/2}} \leq \|(z\mathcal{J} - \mathcal{F}_E)^{-1}\|. \quad (9.27b)$$

The smaller is the value $\text{dist}(z^2, [0, i/2])$, the closer are the lower estimate (9.27b) and the upper estimate (9.27a). However, we would like to estimate of the resolvent in term of the value $\text{dist}(z, \sigma(\mathcal{M}_F))$.

Lemma 9.7. *Let ζ be a point of the spectrum $\sigma(\mathcal{M}_F)$:*

$$\zeta \in \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right], \quad (9.28)$$

and the point z lies on the normal to the interval $\left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right]$ at the point ζ :

$$z = \zeta \pm |z - \zeta| e^{i3\pi/4}. \quad (9.29)$$

Then

$$\text{dist}(z^2, [0, i/2]) = \begin{cases} 2|\zeta| |z - \zeta|, & \text{if } |z - \zeta| \leq |\zeta|, \\ |\zeta|^2 + |z - \zeta|^2 = |z|^2, & \text{if } |z - \zeta| \geq |\zeta|. \end{cases} \quad (9.30)$$

Proof. The condition (9.28) means that $\zeta = \pm|\zeta|e^{i\pi/4}$. Substituting this expression for ζ into (2.8), we obtain

$$z^2 = \pm 2|\zeta||z - \zeta| + i(|\zeta|^2 - |z - \zeta|^2).$$

If $|z - \zeta| \leq |\zeta|$, then the point $i(|\zeta|^2 - |z - \zeta|^2)$ lies on the interval $[0, i/2]$. In this case, $\text{dist}(z^2, [0, i/2]) = 2|\zeta||z - \zeta|$. If $|z - \zeta| \geq |\zeta|$, then the point $i(|\zeta|^2 - |z - \zeta|^2)$ lies on the half-axis $[0, -i\infty)$. In this case,

$$\begin{aligned} \text{dist}(z^2, [0, i/2]) &= \sqrt{(|\zeta|^2 - |z - \zeta|^2)^2 + 4|\zeta|^2|z - \zeta|^2} = \\ &= |\zeta|^2 + |z - \zeta|^2 = |z|^2. \end{aligned}$$

Since $|\zeta|^2 + |z - \zeta|^2 \geq 2|\zeta||z - \zeta|$, in any case, either $|z - \zeta| \leq |\zeta|$, or $|z - \zeta| \geq |\zeta|$, the inequality

$$\text{dist}(z^2, [0, i/2]) \geq 2|\zeta||z - \zeta|. \quad (9.31)$$

holds. \square

Theorem 9.2. *Let ζ be a point of the spectrum $\sigma(\mathcal{M}_F)$ of the operator \mathcal{M}_F , and let the point z lie on the normal to the interval $\sigma(\mathcal{M}_F)$ at the point ζ .*

Then

1. *The resolvent $(z\mathcal{J} - \mathcal{M}_F)^{-1}$ admits the estimate from above:*

$$\|(z\mathcal{J} - \mathcal{M}_F)^{-1}\| \leq A(z) \frac{1}{|\zeta|} \cdot \frac{1}{|z - \zeta|}, \quad (9.32)$$

where $A(z) = \frac{(2|z|^2+1)^{1/2}}{2}$.

2. *If moreover the condition $|z - \zeta| \leq |\zeta|$ is satisfied, then the resolvent $(z\mathcal{J} - \mathcal{M}_F)^{-1}$ also admits the estimate from below:*

$$A(z) \frac{1}{|\zeta|} \cdot \frac{1}{|z - \zeta|} - B(z)|\zeta||z - \zeta| \leq \|(z\mathcal{J} - \mathcal{M}_F)^{-1}\|, \quad (9.33)$$

where $A(z)$ is the same that in (2.9) and $B(z) = \frac{4}{(2|z|^2+1)^{3/2}}$.

3. *For $\zeta = 0$, then the resolvent $(z\mathcal{J} - \mathcal{M}_F)^{-1}$ admits the estimates*

$$2A(z) \frac{1}{|z|^2} - B(z) \leq \|(z\mathcal{J} - \mathcal{M}_F)^{-1}\| \leq 2A(z) \frac{1}{|z|^2}, \quad (9.34)$$

where $A(z)$ and $B(z)$ are the same that in (2.9), (2.10), and z is an arbitrary point of the normal.

In particular, if $\zeta \neq 0$, and z tends to ζ along the normal to the interval $\sigma(\mathcal{M}_F)$, then

$$\|(z\mathcal{J} - \mathcal{M}_F)^{-1}\| = \frac{A(\zeta)}{|\zeta|} \frac{1}{|z - \zeta|} + O(1). \quad (9.35)$$

If $\zeta = 0$ and z tends to ζ along the normal to the interval $\sigma(\mathcal{M}_F)$, then

$$\|(z\mathcal{J} - \mathcal{M}_F)^{-1}\| = |z|^{-2} + O(1), \quad (9.36)$$

where $O(1)$ is a value which remains bounded as z tends to ζ .

Proof. The proof is based on the estimates (9.27) for the resolvent and on Lemma 9.7. Combining the inequality (9.31) with the estimate (9.27a), we obtain the estimate (2.9), which holds for *all* z lying on the normal to the interval $\sigma(\mathcal{M}_F)$ at the point ζ . If moreover z is close enough to ζ , namely the condition $|z - \zeta| \leq |\zeta|$ is satisfied, then the equality holds in (9.31). Combining the equality (9.31) with the estimate (9.27b), we obtain the estimate (2.10).

The asymptotic relation (2.12) is a consequence of the inequalities (2.9) and (2.10) since $\frac{|A(z) - A(\zeta)|}{|z - \zeta|} = O(1)$ as z tends to ζ .

The asymptotic relation (2.14) is a consequence of the inequalities (9.27) and the equality $\text{dist}(z^2, [0, i/2]) = |z|^2$ which holds for all z lying on the normal to the interval $\sigma(\mathcal{M}_F)$ at the point $\zeta = 0$. (See (9.30) for $\zeta = 0$.) \square

Remark 9.1. The estimates (2.9) and (2.10) are formally true also for $\zeta = 0$, but in this case they are not rich in content.

Corollary 9.1. From the asymptotic relations (2.12) and (2.14) it follows that the operator \mathcal{M}_F is not similar to any normal operator. Were the operator \mathcal{M}_F similar to a normal operator \mathcal{N} , the resolvent $(z\mathcal{J} - \mathcal{M}_F)^{-1}$ would admit the estimate

$$\|(z\mathcal{J} - \mathcal{M}_F)^{-1}\| \leq C(\mathcal{N}) \text{dist}(z, \sigma(\mathcal{M}_F)),$$

where $C(\mathcal{N}) < \infty$ is a constant which does not depend on z . However, this estimate is not compatible with the asymptotic relation (2.14).

10. Functional calculus for the operator \mathcal{M}_F .

A direct consequence of the unitary equivalency (9.1) is:

Theorem 10.1.

1. The spectra $\sigma(\mathcal{F}_{\mathbb{R}^+})$ and $\sigma(\mathcal{M}_F)$ coincide:

$$\sigma(\mathcal{F}_{\mathbb{R}^+}) = \sigma(\mathcal{M}_F). \quad (10.1)$$

2. For z out of these spectra, the resolvents are related by the equality

$$(z\mathcal{J} - \mathcal{F}_{\mathbb{R}^+})^{-1} = U^{-1}(z\mathcal{J} - \mathcal{M}_F)^{-1}U. \quad (10.2)$$

The unitary equivalency (9.1) suggests the equality

$$h(\mathcal{F}_{\mathbb{R}^+}) = U^{-1}h(\mathcal{M}_F)U \quad (10.3)$$

for functions h of the operator. The equality (10.3) can be interpreted differently. From (9.1) follows directly that the equality (10.3) holds for any polynomial h . Invoking Theorem 10.1, we conclude that the equality (10.3) holds for every rational function⁵

If h is holomorphic on the spectra $\sigma(\mathcal{F}_{\mathbb{R}^+}) = \sigma(\mathcal{M}_F)$, then the equality (10.3) still holds: $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = U^{-1}h_{\text{hol}}(\mathcal{M}_F)U$, where the operators $h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+})$, $h_{\text{hol}}(\mathcal{M}_F)$ are defined in the sense of the holomorphic functional calculus. To see this, we should multiply the equality (10.2) with $h(z)$ and then integrate the product along a contour which encloses the spectra and is contained in the domain of holomorphy of the function h .

If h is a function which is not holomorphic on the spectra $\sigma(\mathcal{F}_{\mathbb{R}^+}) = \sigma(\mathcal{M}_F)$, then the question “What are $h(\mathcal{F}_{\mathbb{R}^+})$ and $h(\mathcal{M}_F)$?” arises. For a class of functions h which were called $\mathcal{F}_{\mathbb{R}^+}$ -admissible, Definition (2.3), the definition of the operator $h(\mathcal{F}_{\mathbb{R}^+})$ was announced, (2.4). However the existence of the limit in (2.4) is not yet proved, and the properties of the mapping $h \rightarrow h(\mathcal{F}_{\mathbb{R}^+})$ are not yet established.

In fact, we use the equality (10.3) as a *definition* of the operator $h(\mathcal{F}_{\mathbb{R}^+})$ in terms of the operator $h(\mathcal{M}_F)$.

Namely, we first work not with the operator $\mathcal{F}_{\mathbb{R}^+}$, but with its model – the operator \mathcal{M}_F . To work with the operator \mathcal{M}_F is much easier: we can write explicit formulae. So, given an $\mathcal{F}_{\mathbb{R}^+}$ -admissible function h , we first define the operator $h(\mathcal{M}_F)$ by an explicit formula. In particular we justify the passage to the limit analogous to (2.4), where

⁵For a rational function h , the function $h(A)$ of an operator A can be defined directly. For a rational h , all reasonable functional calculuses leads to the same result for $h(A)$.

the resolvent $(z\mathcal{J} - \mathcal{F}_{\mathbb{R}+})^{-1}$ is replaced by the resolvent $(z\mathcal{J} - \mathcal{M}_F)^{-1}$. Then we *transplant* the constructions and the results, related to the model operator \mathcal{M}_F , to the operator $\mathcal{F}_{\mathbb{R}+}$ itself.

Definition 10.1. Let h be a function defined almost everywhere on the spectrum $\sigma(\mathcal{M}_F)$ of the operator \mathcal{M}_F . Then for almost every $\mu \in [0, \infty)$ the function h is defined on the spectrum $\sigma(F(\mu))$ of the matrix $F(\mu)$. Since the matrix $F(\mu)$ is a matrix with simple spectrum, the matrix $h(F(\mu))$ is defined in the sense of elementary functional calculus of matrices. (See Definition 10.3 below).

The function h is said to be \mathcal{M}_F -admissible, if the condition

$$\operatorname{ess\,sup}_{\mu \in [0, \infty)} \|h(F(\mu))\| < \infty, \quad (10.4)$$

is fulfilled.

Definition 10.2. Let h be an \mathcal{M}_F -admissible function. Then the operator $\mathcal{M}_{h(F)}$ is defined according Definition 7.2⁶, and we set

$$h(\mathcal{M}_F) \stackrel{\text{def}}{=} \mathcal{M}_{h(F)}. \quad (10.5)$$

If the function h is a polynomial or a rational function holomorphic on the spectrum $\sigma(\mathcal{M}_F)$, then the operator $h(\mathcal{M}_F)$ can be defined directly, without the formula (10.5). In this case both definition of the operator $h(\mathcal{M}_F)$, the definition (10.5) and the direct one, coincides.

To describe the class of \mathcal{M}_F -admissible functions more explicitly, we have to obtain the explicit expression for the matrix $h(F(\mu))$.

The spectral projectors of the matrix $F(\mu)$.

Calculated the residues $E_{\pm}(\mu)$ at the poles $\zeta_{\pm}(\mu)$ of the the resolvent $(zI - F(\mu))^{-1}$ explicitly, we obtain from (9.13) that

$$E_+(\mu) = \begin{bmatrix} \frac{1}{2} & \frac{f_{-+}(\mu)}{2\zeta(\mu)} \\ \frac{f_{+-}(\mu)}{2\zeta(\mu)} & \frac{1}{2} \end{bmatrix}, \quad E_-(\mu) = \begin{bmatrix} \frac{1}{2} & -\frac{f_{-+}(\mu)}{2\zeta(\mu)} \\ -\frac{f_{+-}(\mu)}{2\zeta(\mu)} & \frac{1}{2} \end{bmatrix}. \quad (10.6)$$

The matrices $E_+(\mu)$, $E_-(\mu)$ satisfy the equalities

$$E_+(\mu)^2 = E_+(\mu), \quad E_-(\mu)^2 = E_-(\mu), \quad (10.7)$$

⁶Definition 7.2 should be applied to the function $G(\mu) = h(F(\mu))$

$$E_+(\mu)E_-(\mu) = E_-(\mu)E_+(\mu) = 0, \quad E_+(\mu) + E_-(\mu) = I, \quad (10.8)$$

$$F(\mu)E_+(\mu) = \zeta_+(\mu)E_+(\mu), \quad F(\mu)E_-(\mu) = \zeta_-(\mu)E_-(\mu). \quad (10.9)$$

The equalities (10.7) means that the matrices $E_+(\mu)$, $E_-(\mu)$ are projectors, the equalities (10.8) means that these projectors are ‘complementary’, the equality (10.9) means that the image subspaces of these projectors are eigensubspaces of the matrix $F(\mu)$. The equalities (10.7), (10.8), (10.9) are special cases of general facts related to root subspaces of a matrix. However these equalities can be obtained directly, using the equality (9.9).

The conjugate numbers $\overline{\zeta_+(\mu)}$, $\overline{\zeta_-(\mu)}$ are the eigenvalues of the hermitian conjugate matrix $F^*(\mu)$. The hermitian conjugate matrices $E_+^*(\mu)$, $E_-^*(\mu)$,

$$E_+^*(\mu) = \begin{bmatrix} \frac{1}{2} & \frac{\overline{f_{+-}(\mu)}}{2\overline{\zeta(\mu)}} \\ \frac{\overline{f_{-+}(\mu)}}{2\overline{\zeta(\mu)}} & \frac{1}{2} \end{bmatrix}, \quad E_-^*(\mu) = \begin{bmatrix} \frac{1}{2} & -\frac{\overline{f_{+-}(\mu)}}{2\overline{\zeta(\mu)}} \\ -\frac{\overline{f_{-+}(\mu)}}{2\overline{\zeta(\mu)}} & \frac{1}{2} \end{bmatrix}, \quad (10.10)$$

are the spectral projectors onto the eigenspaces of the matrix $F^*(\mu)$ corresponding to these eigenvalues:

$$E_+^*(\mu)^2 = E_+^*(\mu), \quad E_-^*(\mu)^2 = E_-^*(\mu), \quad (10.11)$$

$$E_+^*(\mu)E_-^*(\mu) = E_-^*(\mu)E_+^*(\mu) = 0, \quad E_+^*(\mu) + E_-^*(\mu) = I, \quad (10.12)$$

$$F^*(\mu)E_+^*(\mu) = \overline{\zeta_+(\mu)}E_+^*(\mu), \quad F^*(\mu)E_-^*(\mu) = \overline{\zeta_-(\mu)}E_-^*(\mu). \quad (10.13)$$

Functions of the matrix $F(\mu)$.

Definition 10.3. Given $\mu \in [0, \infty)$, let h be a function defined on the spectrum $\sigma(F(\mu))$. Such a function is specified by a pair of numbers $h(\zeta_+(\mu))$ and $h(\zeta_-(\mu))$. Then the matrix $h(F(\mu))$, which is called *the function h of the matrix $F(\mu)$* , is defined as

$$h(F(\mu)) \stackrel{\text{def}}{=} h(\zeta_+(\mu))E_+(\mu) + h(\zeta_-(\mu))E_-(\mu). \quad (10.14)$$

Substituting the expressions (10.6) for $E_+(\mu)$, $E_-(\mu)$ into (10.14), we

obtain

$$\begin{aligned}
h(F(\mu)) &= \\
&= \begin{bmatrix} \frac{h(\zeta_+(\mu)) + h(\zeta_-(\mu))}{2} & \frac{h(\zeta_+(\mu)) - h(\zeta_-(\mu))}{2\zeta(\mu)} f_{+-}(\mu) \\ \frac{h(\zeta_+(\mu)) - h(\zeta_-(\mu))}{2\zeta(\mu)} f_{-+}(\mu) & \frac{h(\zeta_+(\mu)) + h(\zeta_-(\mu))}{2} \end{bmatrix}
\end{aligned} \tag{10.15}$$

The following lemma summarizes the well known facts about functions of matrices.

Lemma 10.1. *It is well known that the correspondence $h(\zeta) \rightarrow h(F(\mu))$ is the algebraic homomorphism of the set of functions defined on the two-point set $\sigma(F(\mu))$ into the set $\mathfrak{M}_{2,2}$ of 2×2 matrices:*

1. *If $h(\zeta) = 1$, that is $h(\zeta_+(\mu)) = 1$, $h(\zeta_-(\mu)) = 1$, then $h(F(\mu)) = I$.*
2. *If $h(\zeta) = \zeta$, that is $h(\zeta_+(\mu)) = \zeta_+(\mu)$, $h(\zeta_-(\mu)) = \zeta_-(\mu)$, then $h(F(\mu)) = F(\mu)$.*
3. *If $h(\zeta) = \alpha_1 h_1(\zeta) + \alpha_2 h_2(\zeta)$, where α_1, α_2 are complex numbers and h_1, h_2 are functions defined on the spectrum $\sigma(F(\mu))$: $h(\zeta_{\pm}(\mu)) = \alpha_1 h_1(\zeta_{\pm}(\mu)) + \alpha_2 h_2(\zeta_{\pm}(\mu))$, then*

$$h(F(\mu)) = \alpha_1 [h_1(F(\mu))] + \alpha_2 h_2(F(\mu)).$$

4. *If $h(\zeta) = h_1(\zeta) h_2(\zeta)$, where h_1, h_2 are functions defined on the spectrum $\sigma(F(\mu))$: $h(\zeta_{\pm}(\mu)) = h_1(\zeta_{\pm}(\mu)) h_2(\zeta_{\pm}(\mu))$, then*

$$h(F(\mu)) = h_1(F(\mu)) h_2(F(\mu)).$$

5. *If $z \notin \sigma(F(\mu))$, that is $z \neq \zeta_{\pm}(\mu)$, and $h(\zeta) = (z - \zeta)$ for $\zeta = \zeta_+(\mu)$ and $\zeta = \zeta_-(\mu)$, then $h(F(\mu)) = (zI - F(\mu))^{-1}$.*
6. *The definition (10.14) of the function h of the matrix $F(\mu)$ is compatible with the holomorphic operator calculus. If h is a function holomorphic on $\sigma(F(\mu))$, then $h(F(\mu)) = h_{\text{hol}}(F(\mu))$*

The properties 1-4 of the correspondence $h \rightarrow h(F(\mu))$ are not evident from the expression (10.15) for $h(F(\mu))$, but are evident from the expression (10.14) and the properties (10.7), (10.8), (10.9) of the matrices $E_+(\mu)$, $E_-(\mu)$.

The norm in the set of functions defined on $\sigma(F(\mu))$.

Let $\mu \in [0, \infty)$. For a function $h(\zeta) = \{h(\zeta_+(\mu)), h(\zeta_-(\mu))\}$, defined on $\sigma(F(\mu))$, we set

$$\|h\|_\mu \stackrel{\text{def}}{=} \left| \frac{h(\zeta_+(\mu)) + h(\zeta_-(\mu))}{2} \right| + \left| \frac{h(\zeta_+(\mu)) - h(\zeta_-(\mu))}{2\zeta(\mu)} \right|. \quad (10.16)$$

It is clear that the mapping $h \rightarrow \|h\|_\mu$ is a norm on the set all functions defined on $\sigma(F(\mu))$.

Lemma 10.2. *If $h = h_1 h_2$, where h_1, h_2 be functions defined on $\sigma(F(\mu))$, then*

$$\|h\|_\mu \leq \|h_1\|_\mu \|h_2\|_\mu. \quad (10.17)$$

Proof. For $j = 1, 2$, let $\frac{h_j(\zeta_+(\mu) + h_j(\zeta_-(\mu)))}{2} = a_j$, $\frac{h_j(\zeta_+(\mu) - h_j(\zeta_-(\mu)))}{2} = b_j$. Then $h_j(\zeta_+(\mu)) = a_j + b_j$, $h_j(\zeta_-(\mu)) = a_j - b_j$, so

$$h(\zeta_+(\mu)) = (a_1 + b_1)(a_2 + b_2), h(\zeta_-(\mu)) = (a_1 - b_1)(a_2 - b_2).$$

The inequality (10.17) takes the form

$$\begin{aligned} \left| \frac{(a_1 + b_1)(a_2 + b_2) + (a_1 - b_1)(a_2 - b_2)}{2} \right| + \left| \frac{(a_1 + b_1)(a_2 + b_2) - (a_1 - b_1)(a_2 - b_2)}{2\zeta(\mu)} \right| &\leq \\ &\leq (|a_1| + |b_1|/|\zeta(\mu)|) (|a_2| + |b_2|/|\zeta(\mu)|), \end{aligned}$$

which can be reduced to the form

$$\begin{aligned} |a_1 a_2 + b_1 b_2| + \left| \frac{a_1 b_2 + b_1 a_2}{\zeta(\mu)} \right| &\leq \\ &\leq \left(|a_1| + \frac{|b_1|}{|\zeta(\mu)|} \right) \left(|a_2| + \frac{|b_2|}{|\zeta(\mu)|} \right). \end{aligned}$$

Since $|\zeta(\mu)| \leq 1$, the last inequality holds for any a_1, b_1, a_2, b_2 . \square

The definition (10.16) of the norm $\|h\|_\mu$ is motivated by the expression (10.15) for the matrix $h(F(\mu))$. It is natural to estimate the norm $\|h(F(\mu))\|$ of the matrix $h(F(\mu))$ in terms of the norm $\|h\|_\mu$ of the function h .

One more estimate for the norm of 2×2 matrix.

Lemma 10.3. *Let*

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad (10.18)$$

be 2×2 matrix. Then the norm of the matrix M , considered as an operator from \mathbb{C}^2 to \mathbb{C}^2 , admits the estimates from above

$$\|M\| \leq \max(|m_{11}| + |m_{12}|, |m_{21}| + |m_{22}|, |m_{11}| + |m_{21}|, |m_{12}| + |m_{22}|), \quad (10.19)$$

and from below

$$\frac{1}{\sqrt{2}} \cdot \max(|m_{11}| + |m_{12}|, |m_{21}| + |m_{22}|, |m_{11}| + |m_{21}|, |m_{12}| + |m_{22}|) \leq \|M\|. \quad (10.20)$$

Proof. The estimate (10.19) is a special case of the Schur estimate. (See for example []). To obtain the estimate (10.20), we apply the matrix M to the vector $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $Mx = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$. For the considered x , the inequality $\|Mx\| \leq \|M\| \cdot \|x\|$ is the inequality $|m_{11}|^2 + |m_{21}|^2 \leq \|M\|^2$. Since $\frac{1}{2} \cdot (|m_{11}| + |m_{21}|)^2 \leq |m_{11}|^2 + |m_{21}|^2$, the inequality $\frac{1}{\sqrt{2}} \cdot (|m_{11}| + |m_{21}|) \leq \|M\|$ holds. Choosing $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we come to the inequality $\frac{1}{\sqrt{2}} \cdot (|m_{12}| + |m_{22}|) \leq \|M\|$. Applying these reasoning to the ajoint matrix M^* and taking into account that $\|M\| = \|M^*\|$, we get the inequalities $\frac{1}{\sqrt{2}} \cdot (|m_{11}| + |m_{12}|) \leq \|M\|$ and $\frac{1}{\sqrt{2}} \cdot (|m_{21}| + |m_{22}|) \leq \|M\|$. The estimate (10.20) summarizes these four inequalities. \square

Estimates of the matrix $h(F(\mu))$.

Lemma 10.4. *Given $\mu \in [0, \infty)$, let $F(\mu)$ be the matrix (8.11), (8.10), and h be a function defined on the spectrum $\sigma(F(\mu)) = \{\zeta_+(\mu), \zeta_-(\mu)\}$ of the matrix $F(\mu)$. Then the matrix $h(F(\mu))$, (10.15), admits the estimates*

$$\frac{1}{2} \|h\|_\mu \leq \|h(F(\mu))\| \leq \|h\|_\mu, \quad (10.21)$$

where $\|h\|_\mu$ is defined by (10.16).

Proof. We apply the estimates (10.19), (10.20), where the matrix which appears in the right hand side of (10.15) is taken as the matrix M . Applying the estimates (10.19), (10.20), we should take into account the estimates (8.17) for $|f_\pm(\mu)|$. \square

Lemma 10.5. *Let $h(\zeta)$ be a function defined almost everywhere on the interval $\sigma(\mathcal{F}_{\mathbb{R}^+}) = \sigma(\mathcal{M}_F)$.*

The function h is $\mathcal{F}_{\mathbb{R}^+}$ -admissible, (Definition 2.3), if and only if it is \mathcal{M}_F -admissible (Definition 10.1).

Proof. In view of (2.16),

$$\|h\|_{\mathcal{F}_{\mathbb{R}^+}} = \operatorname{ess\,sup}_{\mu \in [0, \infty)} \|h\|_{\mu}$$

where $\|h\|_{\mu}$ is defined by (10.16). In view of (10.21),

$$\frac{1}{2} \|h\|_{\mathcal{F}_{\mathbb{R}^+}} \leq \operatorname{ess\,sup}_{\mu \in [0, \infty)} \|h(F(\mu))\| \leq \|h\|_{\mathcal{F}_{\mathbb{R}^+}}. \quad (10.22)$$

□

Now we can *define* the operator $h(\mathcal{F}_{\mathbb{R}^+})$ by the formula (10.3), where the operator $h(\mathcal{M}_F)$ is defined according to Definition 10.2, by the equality (10.5):

Definition 10.4. *Let h be an $\mathcal{F}_{\mathbb{R}^+}$ -admissible function. The operator $h_{\operatorname{mod}}(\mathcal{F}_{\mathbb{R}^+})$ is defined as*

$$h_{\operatorname{mod}}(\mathcal{F}_{\mathbb{R}^+}) \stackrel{\operatorname{def}}{=} U^{-1} \mathcal{M}_{h(F)} U. \quad (10.23)$$

The correspondence $h(\zeta) \rightarrow h_{\operatorname{mod}}(\mathcal{F}_{\mathbb{R}^+})$ is said to be the model-based function calculus for the operator $\mathcal{F}_{\mathbb{R}^+}$.

Theorem 10.2.

1. The mapping $h \rightarrow h_{\operatorname{mod}}(\mathcal{F}_{\mathbb{R}^+})$ is an algebraic homomorphism of the Banach algebra $\mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$ of all $\mathcal{F}_{\mathbb{R}^+}$ -admissible functions into the Banach algebra of all bounded operators in $L^2(\mathbb{R}^+)$.
2. The mapping $h \rightarrow h_{\operatorname{mod}}(\mathcal{F}_{\mathbb{R}^+})$ is a homeomorphism of these Banach algebras. Moreover the two sided estimate

$$\frac{1}{2} \|h\|_{\mathcal{F}_{\mathbb{R}^+}} \leq \|h_{\operatorname{mod}}(\mathcal{F}_{\mathbb{R}^+})\| \leq \|h\|_{\mathcal{F}_{\mathbb{R}^+}} \quad (10.24)$$

holds.

3. Let $\{h_n\}_{1 \leq n < \infty}$ be a sequence of $\mathcal{F}_{\mathbb{R}^+}$ -admissible functions. We assume that:
 - a). The $\mathcal{F}_{\mathbb{R}^+}$ -norms $\|h_n\|_{\mathcal{F}_{\mathbb{R}^+}}$ of these functions are uniformly bounded:

$$\sup_n \|h_n\|_{\mathcal{F}_{\mathbb{R}^+}} < \infty, \quad (10.25)$$

b). For almost every ζ from the interval $\sigma(\mathcal{F}_{\mathbb{R}^+}) = \sigma(\mathcal{M}_F)$, there exists the limit

$$h(\zeta) = \lim_{n \rightarrow \infty} h_n(\zeta). \quad (10.26)$$

Then the function $h(\zeta)$ is $\mathcal{F}_{\mathbb{R}^+}$ -admissible, and

$$h_{\text{mod}}(\mathcal{F}_{\mathbb{R}^+}) = \lim_{n \rightarrow \infty} (h_n)_{\text{mod}}(\mathcal{F}_{\mathbb{R}^+}), \quad (10.27)$$

where the limit is a strong limit of a sequence of operators.

4. The model functional calculus is an extension of the holomorphic functional calculus: if the function h is holomorphic on the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$, then

$$h_{\text{mod}}(\mathcal{F}_{\mathbb{R}^+}) = h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}). \quad (10.28)$$

Proof. The statement 1 of the theorem is a consequence of the properties of the mapping $h(\zeta) \rightarrow h(F(\mu))$ (Lemma 10.1, statements 1-4), of the properties of the mapping $G \rightarrow \mathcal{M}_G$ (Lemma 7.1, Statements 1-3.), and of the equality (9.1).

The Statement 2 of the theorem is a consequence of the estimate (10.22) and of the equality (7.13), applied to $G(\mu) = h(F(\mu))$.

The statement 3 of the theorem is a consequence of Lemma 7.2 and of the following simple fact. Given $\mu \in [0, \infty)$, if the sequences of numbers $h_n(\zeta_+(\mu))$, $h_n(\zeta_-(\mu))$ converge to a numbers $h(\zeta_+(\mu))$, $h(\zeta_-(\mu))$ respectively, then the sequence of matrices $h_n(F(\mu))$ converges to the matrix $h(F(\mu))$.

Now we turn to Statement 4. Let h be a function holomorphic on the spectrum $\sigma(\mathcal{F}_{\mathbb{R}^+})$. We choose a contour γ which encloses the interval $\sigma(\mathcal{F}_{\mathbb{R}^+})$ and is contained in the domain of holomorphy of the function h . By definition,

$$h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = \frac{1}{2\pi i} \int_{\gamma} h(z)(z\mathcal{I} - \mathcal{F}_{\mathbb{R}^+})^{-1} dz.$$

Using (10.2) and (9.3), we rewrite this formula as

$$h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = U^{-1} \left(\frac{1}{2\pi i} \int_{\gamma} h(z) \mathcal{M}_{(zI - F)^{-1}} dz \right) U$$

The operator $\mathcal{M}_{(zI - F)^{-1}}$ is the multiplication operator on the matrix $(zI - F(\mu))^{-1}$ acting in the space \mathcal{K} of functions of the variable μ . The

variable z is a parameter. Therefore we can permute the multiplication with respect to μ and the integration with respect to z :

$$\frac{1}{2\pi i} \int_{\gamma} h(z) \mathcal{M}_{(zI-F)^{-1}} dz = \mathcal{M}_G,$$

where

$$G(\mu) = \frac{1}{2\pi i} \int_{\gamma} h(z) (zI - F(\mu))^{-1} dz.$$

Substituting the expression (9.14) for the resolvent $(zI - F(\mu))^{-1}$ in the last formula, we obtain that

$$G(\mu) = h(\zeta_+(\mu))E_+(\mu) + h(\zeta_-(\mu))E_-(\mu) = h(F(\mu)).$$

So

$$\frac{1}{2\pi i} \int_{\gamma} h(z) \mathcal{M}_{(zI-F)^{-1}} dz = \mathcal{M}_{h(F)}$$

and

$$h_{\text{hol}}(\mathcal{F}_{\mathbb{R}^+}) = U^{-1} \mathcal{M}_{h(F)} U = h_{\text{mod}}(\mathcal{F}_{\mathbb{R}^+}).$$

□

11. Spectral projectors of the operator \mathcal{M}_F .

Explicit expressions for the spectral projectors $E_+(\mu)$, $E_-(\mu)$ of the matrix $F(\mu)$ is done by (10.6).

Lemma 11.1. *The norms of the spectral projectors $E_+(\mu)$, $E_-(\mu)$ of the matrix $F(\mu)$ are:*

$$\|E_+(\mu)\| = \cosh \frac{\pi\mu}{2}, \quad \|E_-(\mu)\| = \cosh \frac{\pi\mu}{2}, \quad 0 < \mu < \infty. \quad (11.1a)$$

In terms of the eigenvalue $\zeta_{\pm}(\mu)$ of the matrix $F(\mu)$,

$$\begin{aligned} \|E_+(\mu)\| &= \frac{1}{2|\zeta_+(\mu)|} \sqrt{1 + 2|\zeta_+(\mu)|^2}, \\ \|E_-(\mu)\| &= \frac{1}{2|\zeta_-(\mu)|} \sqrt{1 + 2|\zeta_-(\mu)|^2}. \end{aligned} \quad (11.1b)$$

Proof. Calculation of the norm of the matrices $E_+(\mu)$, $E_-(\mu)$ can be reduced to calculation of the norms of the appropriate self-adjoint matrices:

$$\|E_+(\mu)\|^2 = \|E_+^*(\mu) E_+(\mu)\|, \quad \|E_-(\mu)\|^2 = \|E_-^*(\mu) E_-(\mu)\|$$

In its turn, calculation of the norm of a selfadjoint matrix can be reduced to calculation of its maximal eigenvalues. The characteristic equation for the matrix $\|E_+^*(\mu) E_+(\mu)\|$ is:

$$p^2 - (\text{trace } E_+^*(\mu) E_+(\mu))p + \det E_+^*(\mu) E_+(\mu) = 0.$$

By direct computation, $\det E_+^*(\mu) E_+(\mu) = |\det E_+(\mu)|^2 = 0$, (recall that $E_+(\mu)$ is a matrix of rank one), and

$$\text{trace } E_+^*(\mu) E_+(\mu) = \frac{1}{4} \left(2 + \frac{|f_{+-}(\mu)|^2}{|\zeta(\mu)|^2} + \frac{|f_{-+}(\mu)|^2}{|\zeta(\mu)|^2} \right).$$

According to (8.10) and (8.16),

$$\frac{|f_{+-}(\mu)|^2}{|\zeta(\mu)|^2} = e^{\mu\pi}, \quad \frac{|f_{-+}(\mu)|^2}{|\zeta(\mu)|^2} = e^{-\mu\pi}. \quad (11.2)$$

Thus,

$$\text{trace } E_+^*(\mu) E_+(\mu) = \frac{1}{4} (2 + e^{\mu\pi} + e^{-\mu\pi}) = \cosh^2 \frac{\mu\pi}{2}.$$

So, the roots of the above characteristic equation are $p = 0$, and $p = \cosh^2 \frac{\mu\pi}{2}$. Therefore, $\|E_+^*(\mu) E_+(\mu)\| = \cosh^2 \frac{\mu\pi}{2}$, and $\|E_+(\mu)\| = \cosh \frac{\mu\pi}{2}$. Analogously, $\|E_-(\mu)\| = \cosh \frac{\mu\pi}{2}$. \square

Theorem 11.1.

1. *The matrices $E_+(\mu)$ and $E_-(\mu)$, which are of rank one, admit the factorizations*

$$E_+(\mu) = u_+(\mu) v_+^*(\mu), \quad E_-(\mu) = u_-(\mu) v_-^*(\mu), \quad (11.3)$$

where

$$u_+(\mu) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \frac{f_{+-}(\mu)}{\zeta(\mu)} \end{bmatrix}, \quad u_-(\mu) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -\frac{f_{+-}(\mu)}{\zeta(\mu)} \end{bmatrix}, \quad (11.4a)$$

$$v_+(\mu) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \frac{f_{-+}(\mu)}{\bar{\zeta}(\mu)} \end{bmatrix}, \quad v_-(\mu) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -\frac{f_{-+}(\mu)}{\bar{\zeta}(\mu)} \end{bmatrix}. \quad (11.4b)$$

2. The vectors $u_+(\mu)$, $u_-(\mu)$ are eigenvectors of the matrix $F(\mu)$:

$$F(\mu)u_+(\mu) = \zeta_+(\mu)u_+(\mu), \quad F(\mu)u_-(\mu) = \zeta_-(\mu)u_-(\mu), \quad (11.5a)$$

The vectors $v_+(\mu)$, $v_-(\mu)$ are eigenvectors of the matrix $F^*(\mu)$:

$$F^*(\mu)v_+(\mu) = \overline{\zeta_+(\mu)}v_+(\mu), \quad F^*(\mu)v_-(\mu) = \overline{\zeta_-(\mu)}v_-(\mu). \quad (11.5b)$$

Proof. The proof is performed by direct calculation. The equality (9.9) is involved in this calculation. \square

Let us calculate the "angle" between the eigenvectors $u_+(\mu)$ and $u_-(\mu)$ of the matrix $F(\mu)$. We recall that if x and y are non-zero vectors of some Hilbert space provided by the scalar product $\langle \cdot, \cdot \rangle$, then the angle $\theta(x, y)$ between x and y is the unique $\theta \in [0, \pi/2]$ such that

$$\cos^2 \theta = \frac{|\langle x, y \rangle|^2}{\langle x, x \rangle \langle y, y \rangle}. \quad (11.6)$$

Lemma 11.2. The angles $\theta(u_+(\mu), u_-(\mu))$ and $\theta(v_+(\mu), v_-(\mu))$ between eigenvectors of the matrices $F(\mu)$ and $F^*(\mu)$ respectively are:

$$\sin \theta(u_+(\mu), u_-(\mu)) = \frac{1}{\cosh \frac{\mu\pi}{2}}, \quad \sin \theta(v_+(\mu), v_-(\mu)) = \frac{1}{\cosh \frac{\mu\pi}{2}}. \quad (11.7)$$

Proof. The proof is performed by direct calculation using the equalities (11.2). \square

Remark 11.1. *The equalities (11.7) can be presented in terms of the eigenvalues $\zeta_+(\mu)$, $\zeta_-(\mu)$ of the matrices $F(\mu)$, $F^*(\mu)$. According to (9.8),*

$$\cosh \frac{\pi\mu}{2} = \frac{\sqrt{1 + 2|\zeta(\mu)|^2}}{2|\zeta(\mu)|}. \quad (11.8)$$

Thus

$$\sin \theta(u_+(\mu), u_-(\mu)) = \frac{2|\zeta(\mu)|}{\sqrt{1 + 2|\zeta(\mu)|^2}}. \quad (11.9)$$

If \mathcal{C} is a contractive operator in a Hilbert space and e_1, e_2 are its eigenvectors corresponding to the eigenvalues ζ_1, ζ_2 , then the angle $\theta(e_1, e_2)$ between the eigenvectors e_1, e_2 admits the estimate from below:

$$\sin \theta(e_1, e_2) \geq \left| \frac{\zeta_1 - \zeta_2}{1 - \zeta_1 \zeta_2} \right|. \quad (11.10)$$

As applied to the contractive matrix $F(\mu)$, the eigenvalues of which are $\zeta_+(\mu) = \zeta(\mu)$, $\zeta_-(\mu) = -\zeta(\mu)$, the estimate (11.10) turns to the inequality

$$\sin \theta(u_+(\mu), u_-(\mu)) \geq \frac{2|\zeta(\mu)|}{1 + |\zeta(\mu)|^2}. \quad (11.11)$$

Comparing (11.9) and (11.11), we see that for small $\zeta(\mu)$ the difference between the actual value of $\sin \theta(u_+(\mu), u_-(\mu))$ and its estimate from below is very small:

$$\begin{aligned} 0 < \sin \theta(u_+(\mu), u_-(\mu)) - \frac{2|\zeta(\mu)|}{1 + |\zeta(\mu)|^2} &= \\ &= \frac{2|\zeta(\mu)|}{\sqrt{1 + 2|\zeta(\mu)|^2}} - \frac{2|\zeta(\mu)|}{1 + |\zeta(\mu)|^2} < |\zeta(\mu)|^5. \end{aligned} \quad (11.12)$$

12. Relation between resolvent-based and model-based functional calculi for the operator $\mathcal{F}_{\mathbb{R}^+}$.

Theorem 12.1. *Let $h(\zeta)$ be an $\mathcal{F}_{\mathbb{R}^+}$ -admissible function: $h(\zeta) \in \mathfrak{B}_{\mathcal{F}_{\mathbb{R}^+}}$.*

Then the operator $\mathcal{M}_{h(F)}$ is representable as a strong limit⁷ of the

⁷In particular, the strong limit in the right hand side of (2.22) exists.

family of integrals over the spectrum $\sigma(\mathcal{M}_F)$:

$$\begin{aligned} \mathcal{M}_{h(F)} &= \\ &= \lim_{\varepsilon \rightarrow +0} \frac{1}{2\pi i} \int_{\sigma(\mathcal{M}_F)} h(s) \left(R_{\mathcal{M}_F}(s - \varepsilon i e^{i\pi/4}) - R_{\mathcal{M}_F}(s + \varepsilon i e^{i\pi/4}) \right) ds, \end{aligned} \quad (12.1)$$

where $R_{\mathcal{M}_F}(z) = (zI - \mathcal{M}_F)^{-1}$ is the resolvent of the operator \mathcal{M}_F , the integral is taken along the interval $\sigma(\mathcal{M}_F) = \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right]$, from the point $-\frac{1}{\sqrt{2}} e^{i\pi/4}$ to the point $\frac{1}{\sqrt{2}} e^{i\pi/4}$.

Proof. Since $R_{\mathcal{M}_F}(z) = \mathcal{M}_{(zI - F)^{-1}}$, the equality holds

$$R_{\mathcal{M}_F}(s - \varepsilon i e^{i\pi/4}) - R_{\mathcal{M}_F}(s + \varepsilon i e^{i\pi/4}) = \mathcal{M}_G,$$

where

$$\begin{aligned} G(\mu; s, \varepsilon) &= \begin{bmatrix} g_{++}(\mu; s, \varepsilon) & g_{+-}(\mu; s, \varepsilon) \\ g_{-+}(\mu; s, \varepsilon) & g_{--}(\mu; s, \varepsilon) \end{bmatrix} = \\ &= \left((s - \varepsilon i e^{i\pi/4})I - F(\mu) \right)^{-1} - \left((s + \varepsilon i e^{i\pi/4})I - F(\mu) \right)^{-1}. \end{aligned} \quad (12.2)$$

(\mathcal{M}_G is the multiplication operator on the matrix function G of the variable μ . The variables s, ε are parameters.) According to the rule (10.15) of calculating of a function of the matrix $F(\mu)$,

$$\begin{aligned} (zI - F(\mu))^{-1} &= \\ &= \begin{bmatrix} \frac{(z - \zeta_+(\mu))^{-1} + (z - \zeta_-(\mu))^{-1}}{2} & \frac{(z - \zeta_+(\mu))^{-1} - (z - \zeta_-(\mu))^{-1}}{2\zeta(\mu)} f_{+-}(\mu) \\ \frac{(z - \zeta_+(\mu))^{-1} - (z - \zeta_-(\mu))^{-1}}{2\zeta(\mu)} f_{-+}(\mu) & \frac{(z - \zeta_+(\mu))^{-1} + (z - \zeta_-(\mu))^{-1}}{2} \end{bmatrix} \end{aligned} \quad (12.3)$$

Substituting the expressions $z = s - \varepsilon i e^{i\pi/4}$ and $z = s + \varepsilon i e^{i\pi/4}$ into (12.3), we obtain from (12.2)

$$\begin{aligned} g_{++}(\mu; s, \varepsilon) &= g_{--}(\mu; s, \varepsilon) = \\ &= \frac{1}{2} \left((s - \varepsilon i e^{i\pi/4} - \zeta_+(\mu))^{-1} + (s - \varepsilon i e^{i\pi/4} - \zeta_-(\mu))^{-1} - \right. \\ &\quad \left. - (s + \varepsilon i e^{i\pi/4} - \zeta_+(\mu))^{-1} - (s + \varepsilon i e^{i\pi/4} - \zeta_-(\mu))^{-1} \right), \end{aligned}$$

$$\begin{aligned}
g_{+-}(\mu; s, \varepsilon) &= \\
&= \frac{1}{2\zeta(\mu)} \left((s - \varepsilon i e^{i\pi/4} - \zeta_+(\mu))^{-1} - (s - \varepsilon i e^{i\pi/4} - \zeta_-(\mu))^{-1} - \right. \\
&\quad \left. - (s + \varepsilon i e^{i\pi/4} - \zeta_+(\mu))^{-1} + (s + \varepsilon i e^{i\pi/4} - \zeta_-(\mu))^{-1} \right) f_{+-}(\mu),
\end{aligned}$$

$$\begin{aligned}
g_{-+}(\mu; s, \varepsilon) &= \\
&= \frac{1}{2\zeta(\mu)} \left((s - \varepsilon i e^{i\pi/4} - \zeta_+(\mu))^{-1} - (s - \varepsilon i e^{i\pi/4} - \zeta_-(\mu))^{-1} - \right. \\
&\quad \left. - (s + \varepsilon i e^{i\pi/4} - \zeta_+(\mu))^{-1} + (s + \varepsilon i e^{i\pi/4} - \zeta_-(\mu))^{-1} \right) f_{-+}(\mu).
\end{aligned}$$

Thus the entries of the matrix in the right hand side of (2.22) take the form $\frac{1}{2\pi i} \int_{\sigma(\mathcal{M}_F)} h(s) g(\mu; s, \varepsilon) ds$, where $g(\mu; s, \varepsilon)$ is one of the four functions $g_{++}(\mu; s, \varepsilon), g_{+-}(\mu; s, \varepsilon), g_{-+}(\mu; s, \varepsilon), g_{--}(\mu; s, \varepsilon)$.

Studying the limiting behavior of the integrals $\frac{1}{2\pi i} \int_{\sigma(\mathcal{M}_F)} h(s) g(\mu; s, \varepsilon) ds$ as $\varepsilon \rightarrow +0$, we can ignore the factors $f_{+-}(\mu), f_{-+}(\mu)$ in the expressions for $g_{+-}(\mu; s, \varepsilon), g_{-+}(\mu; s, \varepsilon)$. These factors depend neither on s , nor on ε . So we can carry these factors out the integrals. We can also ignore the dependence of the values $\zeta_+(\mu), \zeta_-(\mu)$ on the variable μ . We only have to assume that $\zeta_+ = -\zeta_- = \zeta$, where ζ is an arbitrary point of the spectrum $\sigma(\mathcal{M}_F)$. So, we consider the integrals

$$I_p(h; \zeta, \varepsilon) = \frac{1}{2\pi i} \int_{\sigma(\mathcal{M}_F)} h(s) p(\zeta; s, \varepsilon) ds, \quad (12.4a)$$

$$I_q(h; \zeta, \varepsilon) = \frac{1}{2\pi i} \int_{\sigma(\mathcal{M}_F)} h(s) q(\zeta; s, \varepsilon) ds, \quad (12.4b)$$

where

$$\begin{aligned}
p(\zeta; s, \varepsilon) &= \frac{1}{2} \left((s - \varepsilon i e^{i\pi/4} - \zeta)^{-1} + (s - \varepsilon i e^{i\pi/4} + \zeta)^{-1} - \right. \\
&\quad \left. - (s + \varepsilon i e^{i\pi/4} - \zeta)^{-1} - (s + \varepsilon i e^{i\pi/4} + \zeta)^{-1} \right), \\
q(\zeta; s, \varepsilon) &= \frac{1}{2\zeta} \left((s - \varepsilon i e^{i\pi/4} - \zeta)^{-1} - (s - \varepsilon i e^{i\pi/4} + \zeta)^{-1} - \right. \\
&\quad \left. - (s + \varepsilon i e^{i\pi/4} - \zeta)^{-1} + (s + \varepsilon i e^{i\pi/4} + \zeta)^{-1} \right).
\end{aligned}$$

It is clear that

$$g_{++}(\mu; s, \varepsilon) = p(\zeta(\mu); s, \varepsilon), \quad (12.5a)$$

$$g_{--}(\mu; s, \varepsilon) = p(\zeta(\mu); s, \varepsilon), \quad (12.5b)$$

$$g_{+-}(\mu; s, \varepsilon) = q(\zeta(\mu); s, \varepsilon)f_{+-}(\mu), \quad (12.5c)$$

$$g_{-+}(\mu; s, \varepsilon) = q(\zeta(\mu); s, \varepsilon)f_{-+}(\mu). \quad (12.5d)$$

According to Lemma 7.2, the equality 2.22 will be proved if we prove that for almost every $\mu \in [0, \infty)$

$$h(F(\mu)) = \lim_{\varepsilon \rightarrow +0} \begin{bmatrix} I_p(h; \zeta(\mu), \varepsilon) & I_q(h; \zeta(\mu), \varepsilon)f_{+-}(\mu) \\ I_q(h; \zeta(\mu), \varepsilon)f_{-+}(\mu) & I_p(h; \zeta(\mu), \varepsilon) \end{bmatrix}, \quad (12.6)$$

and moreover the matrix functions in the right hand side of (12.6) are bounded for $\mu \in [0, \infty)$ uniformly with respect to $\varepsilon > 0$:

$$\sup_{\mu \in [0, \infty)} \left\| \begin{bmatrix} I_p(h; \zeta(\mu), \varepsilon) & I_q(h; \zeta(\mu), \varepsilon)f_{+-}(\mu) \\ I_q(h; \zeta(\mu), \varepsilon)f_{-+}(\mu) & I_p(h; \zeta(\mu), \varepsilon) \end{bmatrix} \right\| \leq C < \infty$$

for every $\varepsilon > 0$, (12.7)

where $C < \infty$ is a value which does not depend on $\varepsilon > 0$.

The relations (12.6), (12.7) are consequences of the expression (10.15) for $h(F(\mu))$ and the following Lemma:

Lemma 12.1. *Let $h(s)$ be a summable function on the interval $\sigma(\mathcal{M}_F) = \left[-\frac{1}{\sqrt{2}} e^{i\pi/4}, \frac{1}{\sqrt{2}} e^{i\pi/4} \right]$: $\int_{\sigma(\mathcal{M}_F)} |h(s)| |ds| < \infty$.*

1. *If h satisfies the condition*

$$C_p(h) \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{s \in \sigma(\mathcal{M}_F)} \frac{|h(s) + h(-s)|}{2} < \infty, \quad (12.8)$$

then the integrals $I_p(h; \zeta, \varepsilon)$ are bounded for $\zeta \in \sigma(\mathcal{M}_F)$ uniformly with respect to $\varepsilon > 0$:

$$|I_p(h; \zeta, \varepsilon)| \leq C_p(h), \quad \forall \zeta \in \sigma(\mathcal{M}_F), \quad \forall \varepsilon > 0, \quad (12.9)$$

and for almost every $\zeta \in \sigma(\mathcal{M}_F)$ the limiting relation

$$\lim_{\varepsilon \rightarrow +0} I_p(h; \zeta, \varepsilon) = \frac{h(\zeta) + h(-\zeta)}{2} \quad (12.10)$$

holds.

2. If h satisfies the condition

$$C_q(h) \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{s \in \sigma(\mathcal{M}_F)} \frac{|h(s) - h(-s)|}{2|s|} < \infty, \quad (12.11)$$

then the integrals $I_q(h; \zeta, \varepsilon)$ are bounded for $\zeta \in \sigma(\mathcal{M}_F)$ uniformly with respect to $\varepsilon > 0$:

$$|I_q(h; \zeta, \varepsilon)| \leq 4C_q(h), \quad \forall \zeta \in \sigma(\mathcal{M}_F), \quad \forall \varepsilon > 0, \quad (12.12)$$

and for almost every $\zeta \in \sigma(\mathcal{M}_F)$ the limiting relation

$$\lim_{\varepsilon \rightarrow +0} I_q(h; \zeta, \varepsilon) = \frac{h(\zeta) - h(-\zeta)}{2\zeta} \quad (12.13)$$

holds.

The reference to Lemma 12.1 finalizes the proof of Theorem 2.4. \square

Proof of Lemma 12.1. The singular integrals (12.4), which appear in Lemma, are cognate to the Poisson integral. However in (12.4) the integration is performed over the slopping interval $\left[-\frac{1}{\sqrt{2}}e^{i\pi/4}, \frac{1}{\sqrt{2}}e^{i\pi/4}\right]$ and an evaluation point ζ also belongs to this interval. To pass to the more conventional setting, where the interval of integration is real, and the evaluation point is real, we change variables. We put

$$s = e^{i\pi/4}\rho, \quad \zeta = e^{i\pi/4}r, \quad -1/\sqrt{2} \leq r, \rho \leq 1/\sqrt{2}, \quad (12.14a)$$

$$h(e^{i\pi/4}\rho) = H(\rho). \quad (12.14b)$$

Then the integrals (12.4a) takes the form

$$J_P(H; r, \varepsilon) = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} H(\rho) \frac{P(r, \rho; \varepsilon) + P(r, -\rho; \varepsilon)}{2} d\rho, \quad (12.15a)$$

and the integral (12.4b) takes the form

$$J_Q(H; r, \varepsilon) = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} H(\rho) \frac{P(r, \rho; \varepsilon) - P(r, -\rho; \varepsilon)}{2re^{i\pi/4}} d\rho, \quad (12.15b)$$

where $P(r, \rho; \varepsilon)$ is the Poisson kernel:

$$P(r, \rho; \varepsilon) = \frac{1}{\pi} \cdot \frac{\varepsilon}{(r - \rho)^2 + \varepsilon^2}, \quad -\infty < r, \rho < \infty, \quad \varepsilon > 0. \quad (12.16)$$

Splitting the integral in the right hand side of (12.4a) into the sum of two integrals and changing $\rho \rightarrow -\rho$ in the second of these integrals, we transform the expression for $J_P(H; r, \varepsilon)$ to the form

$$J_P(H; r, \varepsilon) = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{H(\rho) + H(-\rho)}{2} P(r, \rho; \varepsilon) d\rho \quad (12.17)$$

Changing $\rho \rightarrow -\rho$ in the integral in (12.4b), we transform the expression for $J_Q(H; r, \varepsilon)$ to the form

$$J_Q(H; r, \varepsilon) = \int_0^{1/\sqrt{2}} \frac{H(\rho) - H(-\rho)}{2\rho e^{i\pi/4}} Q(r, \rho; \varepsilon) d\rho, \quad (12.18)$$

where

$$Q(r, \rho; \varepsilon) = \frac{\rho}{r} (P(r, \rho; \varepsilon) - P(r, -\rho; \varepsilon)),$$

or, in more detail,

$$Q(r, \rho; \varepsilon) = P(|r|, \rho; \varepsilon) \frac{4\rho^2}{(|r| + \rho)^2 + \varepsilon^2}, \quad -\infty < r < \infty, \quad -\infty < \rho < \infty, \quad \varepsilon > 0. \quad (12.19)$$

Since $\frac{4\rho^2}{(|r| + \rho)^2 + \varepsilon^2} \leq 4$ for $\rho > 0$, the inequality

$$0 < Q(r, \rho; \varepsilon) \leq 4P(|r|, \rho; \varepsilon) \quad (12.20)$$

holds for $-\infty < r < \infty$, $0 < \rho < \infty$, $\varepsilon > 0$. Since $\int_{-\infty}^{\infty} P(r, \rho; \varepsilon) d\rho = 1$ for every $r \in (-\infty, \infty)$ and $\varepsilon > 0$, it follows from (12.17) that

$$|J_P(H; r, \varepsilon)| \leq \operatorname{ess\,sup}_{\rho \in [-1/\sqrt{2}, 1/\sqrt{2}]} \frac{|H(\rho) + H(-\rho)|}{2}, \quad -\infty < r < \infty, \quad \varepsilon > 0. \quad (12.21)$$

From (12.18) and (12.20) it follows that

$$|J_Q(H; r, \varepsilon)| \leq 4 \operatorname{ess\,sup}_{\rho \in [-1/\sqrt{2}, 1/\sqrt{2}]} \frac{|H(\rho) - H(-\rho)|}{2|\rho|}, \quad -\infty < r < \infty, \varepsilon > 0. \quad (12.22)$$

Inequality (12.21) is the same that the inequality (12.9), inequality (12.22) is the same that the inequality (12.12).

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a summable function: $\int_{\mathbb{R}} |f(\rho)| d\rho < \infty$. We recall that the point $r \in \mathbb{R}$ is said to be the Lebesgue point for the function f if

$$\lim_{h \rightarrow +0} \frac{1}{h} \int_{r-h}^{r+h} |f(\rho) - f(r)| d\rho = 0. \quad (12.23)$$

The following fact is one of the main facts of the Lebesgue integration theory:

Given a summable function $f : \mathbb{R} \rightarrow \mathbb{R}$, then almost every point $r \in \mathbb{R}$ is a Lebesgue point for the function f .

We also use the following fundamental fact.

Theorem (On limiting behavior of the Poisson integral).

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a summable function, and P be a Poisson kernel, (12.16). Assume that $r, r \in \mathbb{R}$, is a Lebesgue point for f .

Then

$$\lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} P(r, \rho, \varepsilon) f(\rho) d\rho = f(r) \quad (12.24)$$

In particular, the equality (12.24) holds for almost every $r \in \mathbb{R}$.

We apply this theorem to study the limiting behavior of the integral $J_P(H; r, \varepsilon)$, (12.17), as $\varepsilon \rightarrow +0$. This is an integral of the form (12.24), where

$$f(\rho) = 1/2(H(\rho) + H(-\rho)), \quad |\rho| \leq 1/\sqrt{2}; \quad f(\rho) = 0, \quad |\rho| > 1/\sqrt{2}.$$

According to the above mentioned theorem,

$$\lim_{\varepsilon \rightarrow +0} J_P(H; r, \varepsilon) = \frac{H(r) + H(r)}{2} \quad \text{for almost every } r \in [-1/\sqrt{2}, 1/\sqrt{2}]. \quad (12.25)$$

The relation (12.25) is the same that (12.10).

The integral (12.18) is not a Poisson integral, however the study of the integral (12.18) can be reduced to the study of some Poisson integral. From (12.19) it follows that

$$Q(r, \rho; \varepsilon) = P(|r|, \rho; \varepsilon) + T(r, \rho; \varepsilon), \quad (12.26)$$

where the kernel $T(r, \rho; \varepsilon)$ is of the form

$$T(r, \rho; \varepsilon) = P(|r|, \rho; \varepsilon) \frac{(|r| - \rho)(3|r| + \rho) - \varepsilon^2}{(|r| + \rho)^2 + \varepsilon^2}$$

It is clear that

$$|T(r, \rho; \varepsilon)| \leq 3P(|r|, \rho; \varepsilon) \frac{||r| - \rho|}{|r| + \rho} + P(|r|, \rho; \varepsilon) \frac{\varepsilon^2}{r^2} \quad (12.27)$$

From (12.18), (12.26) and (12.27) it follows that

$$J_Q(H; r, \varepsilon) = \int_0^{1/\sqrt{2}} \frac{H(\rho) - H(-\rho)}{2\rho e^{i\pi/4}} P(|r|, \rho; \varepsilon) d\rho + J_T(H; r, \varepsilon), \quad (12.28)$$

where

$$J_T(H; r, \varepsilon) = \int_0^{1/\sqrt{2}} \frac{H(\rho) - H(-\rho)}{2\rho e^{i\pi/4}} T(|r|, \rho; \varepsilon) d\rho,$$

and according to (12.27)

$$\begin{aligned} |J_T(H; r, \varepsilon)| &\leq \operatorname{ess\,sup}_{\rho \in [0, 1/\sqrt{2}]} \frac{|H(\rho) - H(-\rho)|}{2|\rho|} \times \\ &\quad \times 3 \left(\int_{-\infty}^{\infty} P(|r|, \rho; \varepsilon) \frac{||r| - \rho|}{|r| + \rho} d\rho + \frac{\varepsilon^2}{r^2} \right). \end{aligned}$$

For each fixed $r \neq 0$, the function $\varphi(\rho) = \frac{||r| - \rho|}{|r| + \rho}$ is a continuous bounded function of ρ for $\rho \in [-\infty, \infty]$ vanishing at the point $\rho = r$. According to the elementary property of the Poisson integral,

$$\lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} P(|r|, \rho; \varepsilon) \frac{||r| - \rho|}{|r| + \rho} d\rho = 0$$

for every $r \neq 0$. Thus,

$$\lim_{\varepsilon \rightarrow +0} J_T(H; r, \varepsilon) = 0 \quad \text{for every } r \neq 0. \quad (12.29)$$

According to Theorem on limiting behavior of Poisson integral,

$$\lim_{\varepsilon \rightarrow +0} \int_0^{1/\sqrt{2}} \frac{H(\rho) - H(-\rho)}{2\rho e^{i\pi/4}} P(|r|, \rho; \varepsilon) d\rho = \frac{H(r) - H(-r)}{2re^{i\pi/4}} \quad \text{for almost every } r \in [-1/\sqrt{2}, 1/\sqrt{2}]. \quad (12.30)$$

From (12.28), (12.29) and (12.30) it follows that

$$\lim_{\varepsilon \rightarrow +0} J_Q(H; r, \varepsilon) = \frac{H(r) - H(-r)}{2re^{i\pi/4}} \quad \text{for almost every } r \in [-1/\sqrt{2}, 1/\sqrt{2}]. \quad (12.31)$$

The relation (12.31) is the same that the relation (12.13). Lemma 12.1 is proved. \square

Remark 12.1.

1. If the function h is continuous on the interval $[-e^{i\pi/4}, e^{i\pi/4}]$ and vanishes at the endpoints $\zeta = \pm e^{i\pi/4}/\sqrt{2}$ of this interval, then the limit in (12.10) is uniform with respect to ζ . This follows from the elementary properties of the Poisson integral.
2. Let us assume moreover that the function $(h(\zeta) - h(-\zeta))/\zeta$ also is continuous on the interval $[-e^{i\pi/4}, e^{i\pi/4}]$. The integral (12.18) is not a Poisson integral. However the kernel $Q(r, \rho; \varepsilon)$, like the Poisson kernel $P(r, \rho; \varepsilon)$, possesses the properties of an approximative identity. In particular,

$$\int_0^\infty Q(r, \rho; \varepsilon) d\rho = 1 \quad \text{for every } r \in (-\infty, \infty), \varepsilon > 0. \quad (12.32)$$

Therefore the limit in (12.13) is uniform with respect to ζ .

In view of this remark, Theorem 12.1 can be supplemented.

Remark 12.2. Assume that the function h possesses the properties:

1. The function $h(\zeta)$ is continuous on the interval $\left[-\frac{e^{i\pi/4}}{\sqrt{2}}, \frac{e^{i\pi/4}}{\sqrt{2}}\right]$.

2. The function $\frac{h(\zeta)-h(-\zeta)}{\zeta}$ is continuous at the point $\zeta = 0$.

3. $h\left(\pm \frac{e^{i\pi/4}}{\sqrt{2}}\right) = 0$.

Then the limit in (12.1) also exists in the uniform operator topology.

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